

# ANNIHILATORS OF HIGHEST WEIGHT $\mathfrak{sl}(\infty)$ -MODULES

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**ABSTRACT.** We give a criterion for the annihilator in  $U(\mathfrak{sl}(\infty))$  of a simple highest weight  $\mathfrak{sl}(\infty)$ -module to be nonzero. As a consequence we show that, in contrast with the case of  $\mathfrak{sl}(n)$ , the annihilator in  $U(\mathfrak{sl}(\infty))$  of any simple highest weight  $\mathfrak{sl}(\infty)$ -module is integrable, i.e., coincides with the annihilator of an integrable  $\mathfrak{sl}(\infty)$ -module. Furthermore, we define the class of ideal Borel subalgebras of  $\mathfrak{sl}(\infty)$ , and prove that any prime integrable ideal in  $U(\mathfrak{sl}(\infty))$  is the annihilator of a simple  $\mathfrak{b}^0$ -highest weight module, where  $\mathfrak{b}^0$  is any fixed ideal Borel subalgebra of  $\mathfrak{sl}(\infty)$ . This latter result is an analogue of the celebrated Duflo Theorem for primitive ideals.

**Keywords:** primitive ideals, finitary Lie algebras, highest weight modules.

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## 1. INTRODUCTION

The base field is  $\mathbb{C}$ . If  $\mathfrak{g}$  is a semisimple finite-dimensional Lie algebra, the celebrated Duflo Theorem states that any primitive two-sided ideal in the enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$  (i.e., any annihilator of a simple  $U(\mathfrak{g})$ -module) is the annihilator of a simple highest weight  $\mathfrak{g}$ -module.

The purpose of the present paper is to study primitive ideals in the enveloping algebra  $U(\mathfrak{sl}(\infty))$  of the infinite-dimensional Lie algebra  $\mathfrak{sl}(\infty)$ , and in particular to obtain a partial analogue of Duflo's Theorem for  $\mathfrak{sl}(\infty)$ . Recall that the Lie algebra  $\mathfrak{sl}(\infty)$  can be defined in several equivalent ways, for instance as a direct limit  $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$  [Ba1, Ba2, DP1].

The study of two-sided ideals in  $U(\mathfrak{sl}(\infty))$  has been initiated by A. Zhilinskii [Zh1, Zh2, Zh3], and has been continued in [PP]. Zhilinskii's idea has been to study the joint annihilators of certain systems of  $\mathfrak{sl}(n)$ -modules for variable  $n > 2$ , more precisely, the joint annihilators of coherent local systems of finite-dimensional  $\mathfrak{sl}(n)$ -modules as defined in [Zh1]. Zhilinskii has also provided a classification of coherent local systems [Zh1, Zh2]. We call the ideals introduced by Zhilinskii *integrable* (see Section 2.5 for the precise definition).

A corollary of the results in [PP] is that the associated "variety" of an arbitrary ideal in  $U(\mathfrak{sl}(\infty))$  coincides with the associated "variety" of some integrable ideal in  $U(\mathfrak{sl}(\infty))$ . We do not know whether any ideal in  $U(\mathfrak{sl}(\infty))$  is integrable, however in the present paper we prove that the annihilator of any highest weight  $\mathfrak{sl}(\infty)$ -module is an integrable ideal in  $U(\mathfrak{sl}(\infty))$ .

In order to recall the definition of a highest weight  $\mathfrak{sl}(\infty)$ -module, we first need to recall the definition of a splitting Borel subalgebra of  $\mathfrak{sl}(\infty)$ . According to [DP1], a splitting Borel subalgebra is a subalgebra of  $\mathfrak{sl}(\infty)$  which can be obtained as a direct limit of  $\varinjlim_{n \geq 2} \mathfrak{b}_n$  of Borel subalgebras  $\mathfrak{b}_n \subset \mathfrak{sl}(n)$  for a suitable presentation  $\mathfrak{sl}(\infty)$  as a direct limit  $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$ . In contrast with the finite-dimensional case, the splitting Borel subalgebras

of  $\mathfrak{sl}(\infty)$  are not conjugate by the group of automorphisms of  $\mathfrak{sl}(\infty)$ ; in fact, there are uncountably many conjugacy classes (and even isomorphism classes) of splitting Borel subalgebras of  $\mathfrak{sl}(\infty)$ . However, a  $\mathfrak{b}$ -highest weight module is defined as usual as an  $\mathfrak{sl}(\infty)$ -module generated by a 1-dimensional  $\mathfrak{b}$ -submodule.

The difference between the structure of ideals in  $U(\mathfrak{sl}(\infty))$  and in  $U(\mathfrak{g})$  for a finite-dimensional semisimple  $\mathfrak{g}$ , becomes apparent in the fact that the annihilators in  $U(\mathfrak{sl}(\infty))$  of many simple highest weight modules equal to zero. In this paper we give an explicit criterion for a simple  $\mathfrak{b}$ -highest weight module to have nonzero annihilator. A further central result which we establish is that the annihilator of any  $\mathfrak{b}$ -highest weight  $\mathfrak{sl}(\infty)$ -module is integrable.

Our third notable result is an analogue of Duflo's Theorem. We define a special class of splitting Borel subalgebras  $\mathfrak{b}^0 \subset \mathfrak{sl}(\infty)$ , which we call ideal, and prove that any prime integrable ideal in  $U(\mathfrak{sl}(\infty))$  is the annihilator of a simple  $\mathfrak{b}^0$ -highest weight module for any  $\mathfrak{b}^0$ . The ideal Borel subalgebras  $\mathfrak{b}^0$  have the property that the adjoint representation of  $\mathfrak{sl}(\infty)$  is a  $\mathfrak{b}^0$ -highest weight module.

The paper is structured as follows. In Section 2 we review some well known and some not so well known results about the Lie algebra  $\mathfrak{sl}(\infty)$  and its representations. Section 3 contains a precise statement of our main results. The proofs are given in Sections 4, 5 and 6. In Section 7 we characterize simple  $\mathfrak{sl}(\infty)$ -modules which are determined up to isomorphism by their annihilators in  $U(\mathfrak{sl}(\infty))$ , under the assumption that the annihilator is integrable.

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## 2. PRELIMINARIES

**2.1. The Lie algebra  $\mathfrak{sl}(\infty)$ .** The superscript  $*$  indicates dual space, and  $\mathbf{S}^*(\cdot)$  and  $\Lambda^*(\cdot)$  stand respectively for symmetric and exterior algebra. For a Lie algebra  $\mathfrak{g}$ ,  $U(\mathfrak{g})$  stands for the universal enveloping algebra of  $\mathfrak{g}$ . If  $M$  is a  $\mathfrak{g}$ -module, then  $\text{Ann}_{U(\mathfrak{g})} M$  denotes the annihilator of  $M$  in  $U(\mathfrak{g})$ .

The Lie algebra  $\mathfrak{gl}(\infty)$  can be defined as the Lie algebra of matrices  $(a_{ij})_{i,j \in \mathbb{Z}_{>0}}$  each of which has at most finitely many nonzero entries. Equivalently,  $\mathfrak{gl}(\infty)$  can be defined by giving an explicit basis. Let  $\{e_{ij}\}_{i,j \in \mathbb{Z}_{>0}}$  be a basis of a countable-dimensional vector space denoted by  $\mathfrak{gl}(\infty)$ . Set  $\widehat{\mathfrak{h}} := \text{span}\{e_{ii}\}_{i \in \mathbb{Z}_{>0}}$ . The structure of a Lie algebra on  $\mathfrak{gl}(\infty)$  is given by the formula

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj},$$

where  $i, j \in \mathbb{Z}_{>0}$  and  $\delta_{mn}$  is Kronecker's delta.

Next, one defines  $\mathfrak{sl}(\infty)$  as the commutator subalgebra of  $\mathfrak{gl}(\infty)$ :

$$\mathfrak{sl}(\infty) := [\mathfrak{gl}(\infty), \mathfrak{gl}(\infty)].$$

We set

$$\mathfrak{h} := \widehat{\mathfrak{h}} \cap \mathfrak{sl}(\infty).$$

Clearly,  $\widehat{\mathfrak{h}}$  is a maximal commutative subalgebra of  $\mathfrak{gl}(\infty)$ , and  $\mathfrak{h}$  is a maximal commutative subalgebra of  $\mathfrak{sl}(\infty)$ . Moreover,  $\mathfrak{gl}(\infty)$  has the following root decomposition

$$\mathfrak{gl}(\infty) = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{gl}(\infty)^\alpha,$$

similar to the usual root decomposition of  $\mathfrak{gl}(n)$ . Here  $\Delta = \{\varepsilon_i - \varepsilon_j\}_{i,j \in \mathbb{Z}_{>0}}$  where the system of vectors  $\{\varepsilon_j\}_{j \in \mathbb{Z}_{>0}}$  in  $\widehat{\mathfrak{h}}^*$  is dual to the basis  $\{e_{ii}\}_{i \in \mathbb{Z}_{>0}}$  of  $\widehat{\mathfrak{h}}$ . The Lie subalgebra  $\mathfrak{sl}(\infty)$  inherits this root decomposition:

$$\mathfrak{sl}(\infty) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{sl}(\infty)^\alpha,$$

where  $\mathfrak{sl}(\infty)^\alpha = \mathfrak{gl}(\infty)^\alpha$  for  $\alpha \in \Delta$ .

It is not difficult to prove that any Lie algebra obtained as a direct limit  $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$  is isomorphic to  $\mathfrak{sl}(\infty)$  as defined above. Moreover, a general definition of a *splitting Cartan subalgebra*  $\mathfrak{h}'$  of  $\mathfrak{sl}(\infty)$  is as a direct limit of Cartan subalgebras  $\mathfrak{h}'_n$  of  $\mathfrak{sl}(n)$ , where  $\mathfrak{sl}(\infty)$  is identified with  $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$ . Then, as noted in [DPSn], all splitting Cartan subalgebras of  $\mathfrak{sl}(\infty)$  are conjugate via the automorphism group  $\text{Aut } \mathfrak{sl}(\infty)$ . This enables us to henceforth restrict ourselves to considering only the fixed splitting Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{sl}(\infty)$  introduced above.

A *splitting Borel subalgebra* of  $\mathfrak{sl}(\infty) \cong \varinjlim_{n \geq 2} \mathfrak{sl}(n)$  can be defined as a direct limit  $\varinjlim_{n \geq 2} \mathfrak{b}_n$  of Borel subalgebras  $\mathfrak{b}_n \subset \mathfrak{sl}(n)$ , see [DP1]. Since a general splitting Borel subalgebra of  $\mathfrak{sl}(\infty)$  is conjugate under  $\text{Aut}(\mathfrak{sl}(\infty))$  to a splitting Borel subalgebra containing our fixed splitting Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{sl}(\infty)$ , in what follows we only consider splitting Borel subalgebras containing  $\mathfrak{h}$ . The latter Borel subalgebras are given by the following construction. We say that a subset  $\Delta^* \subset \Delta$  is a subset of *positive roots* if

- (1) for any root  $\alpha \in \Delta$ , precisely one of  $\alpha$  and  $-\alpha$  belongs to  $\Delta^*$ ;
- (2)  $\alpha, \beta \in \Delta^*$  and  $\alpha + \beta \in \Delta$  imply  $\alpha + \beta \in \Delta^*$ .

To any positive subset of roots  $\Delta^*$  we assign the Borel subalgebra  $\mathfrak{b}(\Delta^*) := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^*} \mathfrak{sl}(\infty)^\alpha$  of  $\mathfrak{sl}(\infty)$ , and in this

way we obtain all splitting Borel subalgebras of  $\mathfrak{sl}(\infty)$  containing  $\mathfrak{h}$ .

This leads naturally to the observation [DP1] that the splitting Borel subalgebras containing  $\mathfrak{h}$  are in one-to-one correspondence with linear orders on  $\mathbb{Z}_{>0}$ : given such a linear order  $\prec$ , the corresponding subset of positive roots is  $\{\varepsilon_i - \varepsilon_j\}_{i \prec j}$ .

It is easy to see that different Borel subalgebras containing  $\mathfrak{h}$  do not have to be  $\text{Aut } \mathfrak{sl}(\infty)$ -conjugate, as they simply may not be isomorphic as abstract Lie algebras. Consider, for instance, the following three linear orders on  $\mathbb{Z}_{>0}$ :

- (i)  $\dots \prec 5 \prec 3 \prec 1 \prec 2 \prec 4 \prec 6 \prec \dots$ ,

- (ii)  $1 \prec 2 \prec 3 \prec \dots$ ,
- (iii)  $1 \prec 3 \prec 5 \prec \dots \prec 2n+1 \prec \dots \prec 2n \prec \dots \prec 4 \prec 2$ .

The reader can check that the corresponding Borel subalgebras are not isomorphic as Lie algebras.

**2.2.  $S$ -notation.** Let  $S$  be a subset of  $\mathbb{Z}_{>0}$ . We denote by  $\mathfrak{sl}(S)$  the subalgebra of  $\mathfrak{sl}(\infty)$  spanned by

$$\{e_{ij}\}_{i,j \in S, i \neq j} \quad \text{and} \quad \{e_{ii} - e_{jj}\}_{i,j \in S}.$$

Then  $\mathfrak{sl}(\mathbb{Z}_{>0}) = \mathfrak{sl}(\infty)$ .

Set  $\mathfrak{h}_S := \mathfrak{h} \cap \mathfrak{sl}(S)$ . Note that

(1) if  $S$  is finite, then  $\mathfrak{sl}(S)$  is isomorphic to  $\mathfrak{sl}(n)$  where  $n = |S|$  is the cardinality of  $S$ , and  $\mathfrak{h}_S$  is a Cartan subalgebra of  $\mathfrak{sl}(S)$ ;

(2) if  $S$  is infinite, then  $\mathfrak{sl}(S)$  is isomorphic to  $\mathfrak{sl}(\infty)$ , and  $\mathfrak{h}_S$  is a splitting Cartan subalgebra of  $\mathfrak{sl}(S)$ .

Next, we fix a splitting Borel subalgebra  $\mathfrak{b} \supset \mathfrak{h}$  of  $\mathfrak{sl}(\infty)$  and put  $\mathfrak{b}_S := \mathfrak{sl}(S) \cap \mathfrak{b}$ . We note that

(1) if  $S$  is finite, then  $\mathfrak{b}_S$  is a Borel subalgebra of  $\mathfrak{sl}(S)$ ,

(2) if  $S$  is infinite, then  $\mathfrak{b}_S$  is a splitting Borel subalgebra of  $\mathfrak{sl}(S)$ .

Let  $\mathbb{C}^S$  denote the set of functions from  $S$  to  $\mathbb{C}$ . Clearly,  $\mathbb{C}^S$  is a vector space of dimension  $|S|$ . When  $S = \{1, \dots, n\}$  we write simply  $\mathbb{C}^n$  instead of  $\mathbb{C}^{\{1, \dots, n\}}$ . There is a surjective homomorphism from  $\mathbb{C}^S$  to  $\mathfrak{h}_S^*$ :

$$(1) \quad f \mapsto \lambda_f, \quad \lambda_f(e_{ii} - e_{jj}) = f(i) - f(j).$$

For any  $f \in \mathbb{C}^S$  we denote by  $|f|$  the cardinality of the image of  $f$ . A weight  $\lambda \in \mathfrak{h}_S^*$  is  $\mathfrak{sl}(S)$ -integral, or simply *integral*, if  $\lambda(e_{ii} - e_{jj}) \in \mathbb{Z}$  for all  $i, j \in S$ . Respectively, a function  $f \in \mathbb{C}^S$  is *integral* if  $f(i) - f(j) \in \mathbb{Z}$  for all  $i, j \in S$ . A function is *almost integral* if there exists a finite subset  $F \subset S$  such that  $f|_{S \setminus F}$  is integral.

If  $\mathfrak{b}_S \supset \mathfrak{h}_S$  is a fixed splitting Borel subalgebra of  $\mathfrak{sl}(S)$ , then an integral weight  $\lambda \in \mathfrak{h}_S^*$  is  $\mathfrak{b}_S$ -dominant if  $\lambda(e_{ii} - e_{jj}) \geq 0$  for  $i \prec j$  where the order  $\prec$  on  $S$  is determined by  $\mathfrak{b}_S$ . Respectively, an integral function  $f \in \mathbb{C}^S$  is  $\prec$ -dominant if  $f(i) - f(j) \geq 0$  for  $i \prec j$ .

Let  $\prec$  be a linear order on  $S$ , and let  $S = S_1 \sqcup \dots \sqcup S_t$  be a finite partition of  $S$ . We say that the partition  $\{S_i\}_{i \leq t}$  is *compatible* with the order  $\prec$  if

$$i_0 \prec j_0 \Leftrightarrow i < j$$

for any  $i \neq j \leq t$  and any  $i_0 \in S_i, j_0 \in S_j$ . Finally, we say that  $f \in \mathbb{C}^S$  is *locally constant with respect to  $\prec$*  if there exists a compatible partition  $S_1 \sqcup \dots \sqcup S_t$  of  $S$ , such that  $f$  is constant on  $S_i$  for any  $i \leq t$ .

We call a splitting Borel subalgebra  $\mathfrak{b}_S \supset \mathfrak{h}_S$  of  $\mathfrak{sl}(S)$  *ideal* if there is a partition  $S = S_1 \sqcup S_2 \sqcup S_3$ , compatible with the order  $\prec$  defined by  $\mathfrak{b}_S$ , such that

(a)  $S_1$  is countable and  $\prec$  restricted to  $S_1$  is isomorphic to the standard order on  $\mathbb{Z}_{>0}$ ,

(b)  $S_3$  is countable and  $\prec$  restricted to  $S_3$  is isomorphic to the standard order on  $\mathbb{Z}_{<0}$

( $S_2$  may be empty). Clearly the Borel subalgebra defined by the above order (iii) is ideal, while the Borel subalgebras defined by (i) and (ii) are not ideal.

**2.3. Highest weight  $\mathfrak{sl}(S)$ -modules.** Fix a splitting Borel subalgebra  $\mathfrak{b}_S$  of  $\mathfrak{sl}(S)$ , corresponding to a linear order  $\prec$  on  $S$ . A *Verma module* is defined as an induced module

$$M_{\mathfrak{b}_S}(f) := U(\mathfrak{sl}(S)) \otimes_{U(\mathfrak{b}_S)} \mathbb{C}_f,$$

where  $\mathbb{C}_f$  is a one-dimensional  $\mathfrak{b}_S$ -module determined by a weight  $\lambda_f \in \mathfrak{h}_S^*$ . By definition, a  $\mathfrak{b}_S$ -highest weight module is an  $\mathfrak{sl}(\infty)$ -module isomorphic to a quotient of  $M_{\mathfrak{b}_S}(f)$ . It is not difficult to prove that  $M_{\mathfrak{b}_S}(f)$  has a unique simple quotient  $L_{\mathfrak{b}_S}(f)$ , see [DP1].

As  $S$  and  $\mathfrak{b}_S$  are fixed, in the rest of Section 2.3 we write simply  $M(f)$  and  $L(f)$  instead of  $M_{\mathfrak{b}_S}(f)$  and  $L_{\mathfrak{b}_S}(f)$ . We fix also a function  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  and a highest weight vector  $v$  of  $L(f)$ . For any subset  $S' \subset S$  we denote by  $\widehat{L}(f|_{S'})$  the  $\mathfrak{sl}(S')$ -submodule of  $L(f)$  generated by  $v$ . Obviously  $\widehat{L}(f|_{S'})$  is a quotient of  $M(f|_{S'})$ , and  $L(f|_{S'})$  is the unique simple quotient of  $\widehat{L}(f|_{S'})$ .

For any finite subset  $F \subset S$ , let  $w_F$  be a fixed highest weight vector in  $M(f|_F)$ , and let  $v_F$  be its image in  $L(f|_F)$ . Let  $F \subset F' \subset S$  be two finite subsets. Then there exists a unique morphism of  $\mathfrak{sl}(F)$ -modules

$$\psi_{F, F'} : M(f|_F) \rightarrow L(f|_{F'})$$

such that  $w_F \mapsto v_{F'}$ . It is clear that if  $F'' \supset F' \supset F$  then  $\ker \psi_{F, F''} \subset \ker \psi_{F, F'}$ . Since the  $\mathfrak{sl}(F)$ -module  $M(f|_F)$  has finite length, there exists a sufficiently large finite set  $\bar{F} \supset F$  such that  $\ker \psi_{F, \bar{F}} \subset \ker \psi_{F', F}$  for any finite set  $F' \supset F$ . We put  $\psi_F := \psi_{F, \bar{F}}$ .

**Proposition 2.1.** The  $\mathfrak{sl}(F)$ -module  $\widehat{L}(f|_F)$  is isomorphic to the image of  $\psi_F$ .

*Proof.* Let  $F \subset F'$  be two finite subsets of  $S$ . There exists a finite subset  $\bar{F} \subset S$  such that  $F \subset \bar{F}$ ,  $\ker \psi_{F, \bar{F}} = \ker \psi_F$ , and  $\ker \psi_{F', \bar{F}} = \ker \psi_{F'}$ . Then  $\text{im } \psi_{F, \bar{F}}$  is isomorphic to  $\text{im } \psi_F$  and is equal to  $U(\mathfrak{sl}(F)) \cdot v_{\bar{F}}$ , and  $\text{im } \psi_{F', \bar{F}}$  is isomorphic to  $\text{im } \psi_{F'}$  and is equal to  $U(\mathfrak{sl}(F')) \cdot v_{\bar{F}}$ . This defines an embedding of  $\text{im } \psi_F$  to  $\text{im } \psi_{F'}$  such that  $\psi_F(w_F) \mapsto \psi_{F'}(w_{F'})$ .

The limit of the direct system of such morphisms over all finite subsets  $F$  of  $S$  defines an  $\mathfrak{sl}(\infty)$ -module  $\tilde{L}(f)$ . Clearly, the direct limit of the vectors  $\psi_F(w_F)$  is a highest weight vector of weight  $\lambda_f$  in  $\tilde{L}(f)$ . Denote this vector by  $\tilde{v}$ . We claim that  $\tilde{L}(f)$  is isomorphic to  $L(f)$ . For the proof we provide two  $\mathfrak{sl}(\infty)$ -morphisms  $L(f) \rightarrow \tilde{L}(f)$  and  $\tilde{L}(f) \rightarrow L(f)$  such that  $v \mapsto \tilde{v}$  and  $\tilde{v} \mapsto v$  respectively.

The morphism  $\tilde{L}(f) \rightarrow L(f)$  arises from the fact that  $\tilde{L}(f)$  is a highest weight module with highest weight  $\lambda_f$ . We may assume that under this morphism  $\tilde{v}$  goes to  $v$  (in general,  $\tilde{v}$  maps to some vector proportional to  $v$ ). Now we construct a morphism  $L(f) \rightarrow \tilde{L}(f)$ . For any set  $F$  we pick  $\bar{F}$  as described above and consider the chain

$$M(f|_F) \rightarrow \hat{L}(f|_F) \rightarrow \hat{L}(f|_{\bar{F}}) \rightarrow L(f|_{\bar{F}})$$

of  $\mathfrak{sl}(F)$ -morphisms whose composition is  $\psi_F$ . This defines an  $\mathfrak{sl}(F)$ -morphism  $\hat{L}(f|_F) \rightarrow \text{im } \psi_F$ . By passing to the direct limit, we obtain the desired  $\mathfrak{sl}(\infty)$ -morphism  $L(f) \rightarrow \tilde{L}(f)$ .

Since the  $\mathfrak{sl}(F)$ -submodule of  $\tilde{L}(f)$  generated by the image of  $v$  in  $\tilde{L}(f)$  is isomorphic to  $\text{im } \psi_F$ , the proposition is proved.  $\square$

Any compatible partition  $S_1 \sqcup \dots \sqcup S_t$  of  $S$  defines a parabolic subalgebra of  $\mathfrak{sl}(S)$ : this is the algebra  $\mathfrak{p}$  with root decomposition

$$\mathfrak{h}_S \oplus \bigoplus_{i \prec j \text{ or } i \neq j \in S_k \text{ for some } k} \mathbb{C} e_{ij}.$$

We set  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ , where

$$\mathfrak{l} := \mathfrak{h}_S + \bigoplus_{i \in S_k} \mathfrak{sl}(S_i), \quad \mathfrak{n} := \bigoplus_{e_{ij} \notin \mathfrak{l}, i \prec j} \mathbb{C} e_{ij}.$$

Set also  $\mathfrak{n}^- := \bigoplus_{e_{ij} \notin \mathfrak{l}, j \prec i} \mathbb{C} e_{ij}$ .

**Proposition 2.2.** Let  $S_1 \sqcup \dots \sqcup S_t$  be a compatible partition of  $S$ , and  $f \in \mathbb{C}^S$  be a function such that

$$f(k') - f(l') \notin \mathbb{Z}$$

for all  $k' \in S_i, l' \in S_l$  where  $i < l$ . Then  $L(f)$  is isomorphic to

$$U(\mathfrak{sl}(S)) \otimes_{U(\mathfrak{p})} L(f)^{\mathfrak{n}},$$

where  $L(f)^{\mathfrak{n}}$  stands for the  $\mathfrak{n}$ -invariants of  $L(f)$ . Moreover, as an  $\mathfrak{sl}(S_1) \oplus \dots \oplus \mathfrak{sl}(S_t)$ -module,  $L(f)^{\mathfrak{n}}$  is isomorphic to

$$L(f|_{S_1}) \otimes \dots \otimes L(f|_{S_t}).$$

*Proof.* We set  $L_{\mathfrak{p}} := U(\mathfrak{l}) \cdot v$ . Standard arguments show that  $L_{\mathfrak{p}}$  is a simple  $\mathfrak{l}$ -module and that  $L_{\mathfrak{p}} = L(f)^{\mathfrak{n}}$ . Therefore we have a natural surjective  $\mathfrak{sl}(S)$ -morphism

$$\alpha : U(\mathfrak{sl}(S)) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}} \rightarrow L(f).$$

We claim that  $\alpha$  is an isomorphism. For this it suffices to show that

$$\beta : U(\mathfrak{n}^-) \otimes_{\mathbb{C}} L_{\mathfrak{p}} \rightarrow L(f) \quad (n \otimes l \mapsto nl)$$

is injective. However, the injectivity of  $\beta$  follows from the fact established above that the natural map  $\hat{L}(f|_F) \rightarrow L(f|_{\bar{F}})$  is an injection for any finite subset  $F \subset S$ .

Next, one notes that the simplicity of  $L_{\mathfrak{p}}$  as a  $\mathfrak{l}$ -module implies its simplicity as an  $[\mathfrak{l}, \mathfrak{l}]$ -module. This follows from the fact that any  $\mathfrak{h}$ -weight space of  $L_{\mathfrak{p}}$  is also an  $(\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}])$ -weight space. Then, since  $L(f|_{S_1}) \otimes \dots \otimes L(f|_{S_t})$  and  $L_{\mathfrak{p}}$  are simple  $([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{b}_S)$ -highest weight  $(\mathfrak{sl}(S_1) \oplus \dots \oplus \mathfrak{sl}(S_t) = [\mathfrak{l}, \mathfrak{l}])$ -modules with the same highest weight, they are isomorphic.  $\square$

We say that an  $\mathfrak{sl}(S)$ -module  $M$  is *integrable*, if

$$\dim(U(\mathfrak{g}) \cdot m) < \infty$$

for any  $m \in M$  and any finite-dimensional Lie subalgebra  $\mathfrak{g} \subset \mathfrak{sl}(S)$ .

**Proposition 2.3.** Let  $S_1 \sqcup \dots \sqcup S_t$  be a partition of  $S$  compatible with  $\prec$ . Then the  $\mathfrak{sl}(S)$ -module  $L(f)$  is  $\mathfrak{sl}(S_k)$ -integrable if and only if  $f|_{S_k} \in \mathbb{C}^{S_k}$  is dominant.

*Proof.* Assume that  $L(f)$  is  $\mathfrak{sl}(S_k)$ -integrable. Then the  $(\mathfrak{sl}(2) \cong \mathfrak{sl}(\{i, j\}))$ -module  $U(\mathfrak{sl}(\{i, j\})) \cdot v$  is integrable for  $i, j \in S_k$ . Hence  $f(i) - f(j) \in \mathbb{Z}$  for  $i, j \in S_k$  by a well-known statement about  $\mathfrak{sl}(2)$ .

Now we wish to prove that if  $f|_{S_k}$  is dominant, then  $L(f)$  is  $\mathfrak{sl}(S_k)$ -integrable. Clearly, it suffices to show that  $\widehat{L}(f|_F)$  is  $\mathfrak{sl}(S_k \cap S)$ -integrable for any finite subset  $F \subset S$ . According to Proposition 2.1,  $\widehat{L}(f|_F) \cong \text{im } \psi_F$ . The fact that  $\text{im } \psi_F$  is  $\mathfrak{sl}(S_k \cap S)$ -integrable follows from the well-known fact, concerning modules over finite-dimensional Lie algebras, that, for any finite subset  $F' \subset S$ , the  $\mathfrak{sl}(F')$ -module  $L(f|_{F'})$  is  $\mathfrak{sl}(S_k \cap F')$ -integrable.  $\square$

**Corollary 2.4.** The  $\mathfrak{sl}(S)$ -module  $L(f)$  is integrable if and only if  $f \in \mathbb{C}^S$  is dominant.

**Corollary 2.5.** Assume that  $f$  is locally constant with respect to a compatible partition  $S_1 \sqcup \dots \sqcup S_n$  of  $S$ . Then  $L(f)$  is an integrable  $\mathfrak{sl}(S_i)$ -module for any  $i \leq n$ .

**2.4. Ideals of  $U(\mathfrak{sl}(\infty))$ .** Let  $I$  be an ideal of  $U(\mathfrak{sl}(\infty))$ . Under an ideal we always mean a two-sided ideal. Fix an exhaustion

$$(2) \quad \mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(3) \hookrightarrow \dots \hookrightarrow \mathfrak{sl}(n) \hookrightarrow \mathfrak{sl}(n+1) \hookrightarrow \dots$$

of  $\mathfrak{sl}(\infty)$ . Then  $I = \varinjlim_{n \geq 2} (I \cap U(\mathfrak{sl}(n)))$ . Set  $I_n := I \cap U(\mathfrak{sl}(n))$ . Let  $\text{Var } I_n \subset \mathfrak{sl}(n)^*$  be the associated variety of  $I_n$ . By identifying  $\mathfrak{sl}(n)$  and  $\mathfrak{sl}(n)^*$  via the Killing form we can assume that  $\text{Var } I_n \subset \mathfrak{sl}(n)$ .

For any positive integer  $r$  we introduce the varieties

$$\mathfrak{sl}(n)^{\leq r} := \{x \in \mathfrak{sl}(n) \mid \exists \lambda \in \mathbb{C} \text{ such that } \text{rk}(x - \lambda) \leq r\},$$

where  $\lambda$  is understood as a scalar  $n \times n$ -matrix. One can easily see that  $\mathfrak{sl}(n)^{\leq r}$  is an  $SL(n)$ -stable subvariety of  $\mathfrak{sl}(n)$ .

The following theorem reproduces the claim of [PP, Corollary 6.2 b)] for  $\mathfrak{sl}(\infty)$ .

**Theorem 2.6.** For any nonzero ideal  $I \subset U(\mathfrak{sl}(\infty))$  such that  $\text{Var } I_n \neq 0$  for some  $n$ , there exists a positive integer  $r$  such that  $\text{Var } I_n = \mathfrak{sl}(n)^{\leq r}$  for any  $n \geq 2$ .

**2.5. Integrable ideals and coherent local systems.** We say that an ideal  $I \subset U(\mathfrak{sl}(\infty))$  is *integrable*, if  $I$  is the annihilator of an integrable  $\mathfrak{sl}(\infty)$ -module. Integrable ideals are closely connected with coherent local systems of modules which we define next.

Let  $\text{Irr}_n$  denote the set of isomorphism classes of simple finite-dimensional  $\mathfrak{sl}(n)$ -modules.

**Definition 2.7.** A *coherent local system of modules* (further shortened as *c.l.s.*) for  $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$  is a collection of subsets

$$\{Q_n\}_{n \in \mathbb{Z}_{\geq 2}} \subset \prod_{n \in \mathbb{Z}_{\geq 2}} \text{Irr}_n$$

such that  $Q_m = \langle Q_n \rangle_m$  for any  $n > m$ , where  $\langle Q_n \rangle_m$  denotes the set of isomorphism classes of all simple  $\mathfrak{sl}(m)$ -constituents of the  $\mathfrak{sl}(n)$ -modules from  $Q_n$ .

A. Zhilinskii [Zh2, Zh3] has classified c.l.s. for  $\mathfrak{sl}(\infty)$  and more generally for any locally simple Lie algebra. Moreover, if  $Q$  is a c.l.s., then

$$I(Q_m) := \cap_{z \in Q_m} \text{Ann}_{U(\mathfrak{sl}(m))} z \subset \cap_{z \in Q_n} \text{Ann}_{U(\mathfrak{sl}(n))} z =: I(Q_n)$$

for any  $n > m$ . Therefore  $\cup_n (\cap_{z \in Q_n} \text{Ann}_{U(\mathfrak{sl}(n))} z) = \cup_n I(Q_n)$  is an ideal of  $U(\mathfrak{sl}(\infty))$ ; we denote it by  $I(Q)$ . It follows from [Zh2, Lemma 1.1.2] that  $I(Q)$  is integrable.

It turns out that Zhilinskii's classification of c.l.s. yields a classification of integrable ideals of  $U(\mathfrak{sl}(\infty))$ . In this paper we present only the classification of c.l.s. For the classification of integrable ideals see [PP, Theorem 7.9].

A c.l.s.  $Q$  is *irreducible* if  $Q \neq Q' \cup Q''$  with  $Q' \not\subset Q''$  and  $Q'' \not\subset Q'$ . The following proposition clarifies the role of the irreducible c.l.s.

**Proposition 2.8.** a) If  $Q$  is an irreducible c.l.s., then  $I(Q)$  is the annihilator of a simple  $\mathfrak{sl}(\infty)$ -module. In particular,  $I(Q)$  is primitive and hence prime.

b) If  $I$  is an integrable prime ideal of  $U(\mathfrak{sl}(\infty))$ , then  $I = I(Q)$  for an irreducible c.l.s.  $Q$ .

*Proof.* Part a) follows directly from [Zh2, Lemma 1.1.2]. Part b) is a consequence of [PP, Theorem 7.9].  $\square$

Fix  $n$ . The set  $\text{Irr}_n$  is parametrized by the lattice of integral dominant weights of  $\mathfrak{sl}(n)$ . Let  $z_1, z_2$  be isomorphism classes of simple  $\mathfrak{sl}(n)$ -modules with respective highest weights  $\lambda_1, \lambda_2$ . We denote by  $z_1 z_2$  the isomorphism class of the simple module with highest weight  $\lambda_1 + \lambda_2$ . If  $S_1, S_2 \subset \text{Irr}_n$  we set

$$S_1 S_2 := \{z \in \text{Irr}_n \mid z = z_1 z_2 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

If  $Q', Q''$  are c.l.s., we denote by  $Q' Q''$  the smallest c.l.s. such that  $(Q')_i (Q'')_i \subset (Q' Q'')_i$ . By [Zh2]

$$(Q')_n(Q'')_n = (Q'Q'')_n.$$

2.5.1. *Zhilinskii's classification of c.l.s.* In this subsection we reproduce Zhilinskii's classification of c.l.s. for  $\mathfrak{sl}(\infty)$  [Zh1]. Any integrable  $\mathfrak{sl}(\infty)$ -module  $M$  determines a c.l.s.  $Q := \{Q_n\}_{n \in \mathbb{Z}_{>0}}$ , where

$$Q_n := \{z \in \text{Irr}_n \mid \text{Hom}(z, M) \neq 0\}.$$

We set  $\text{cls}(M) := Q$ . Moreover,  $\text{Ann}_{\text{U}(\mathfrak{sl}(\infty))} M = I(\text{cls}(M))$ . We construct an irreducible c.l.s. as the c.l.s. of some explicitly given integrable  $\mathfrak{sl}(\infty)$ -module.

Let  $V(\infty)$  denote a vector space with basis  $\{e_i\}_{i \in \mathbb{Z}_{>0}}$ . We endow  $V(\infty)$  with an action of  $\mathfrak{sl}(\infty)$  by putting

$$e_{ij} \cdot e_k = e_i \delta_{jk}, \quad (e_{ii} - e_{jj}) \cdot e_k = e_i \delta_{ik} - e_j \delta_{jk} \quad \text{for } i, j, k \in \mathbb{Z}_{>0}.$$

In this way  $V(\infty)$  becomes a simple integrable  $\mathfrak{sl}(\infty)$ -module, and we call it the *natural  $\mathfrak{sl}(\infty)$ -module*. By  $V(\infty)_*$  we denote the restricted dual to  $V(\infty)$ , i.e., the  $\mathfrak{sl}(\infty)$ -submodule of  $V(\infty)^*$  spanned by the vectors  $\{e_i^*\}_{i \in \mathbb{Z}_{>0}}$  which satisfy

$$e_i^*(e_j) = \delta_{ij}.$$

Any irreducible c.l.s.  $Q$  for  $\mathfrak{sl}(\infty)$  is a product of the following *basic* c.l.s.:

$$\begin{aligned} \mathcal{E} &:= \text{cls}(\Lambda^1 V(\infty)), \quad \mathcal{L}_p := \text{cls}(\Lambda^p V(\infty)), \quad \mathcal{L}_p^\infty := \text{cls}(\mathbf{S}^p(V(\infty) \otimes \mathbb{C}^p)), \\ \mathcal{R}_q &:= \text{cls}(\Lambda^q V(\infty)_*), \quad \mathcal{R}_q^\infty := \text{cls}(\mathbf{S}^q(V(\infty)_* \otimes \mathbb{C}^q)), \quad \mathcal{E}^\infty \text{ (all modules)}, \end{aligned}$$

where  $p, q \in \mathbb{Z}_{\geq 0}$ . More precisely, any irreducible c.l.s. is expressed uniquely as

$$(3) \quad (\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} \dots \mathcal{R}_l^{z_l}),$$

where

$$m, n, l, v, w, x_i, z_j \in \mathbb{Z}_{\geq 0}.$$

Here, if  $v = 0$  (respectively  $w = 0$ ), then  $\mathcal{L}_v^\infty$  (respectively  $\mathcal{R}_w^\infty$ ) is assumed to be the identity (the c.l.s. consisting of the isomorphism class of the trivial 1-dimensional module at all levels). In [Zh2] the above formulas are called the *unique factorization property*.

2.6. **C.l.s. of simple integrable highest weight modules.** We start with the following definition.

**Definition 2.9.** A c.l.s.  $Q$  is of *finite type* if  $Q_n$  is finite for any  $n$ .

One can easily check that the irreducible c.l.s. of finite type are precisely the c.l.s. of the form (3) with  $v = w = 0$ .

Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be an integral function. We assume that a linear order on  $\mathbb{Z}_{>0}$  is fixed and therefore we use the notations of Section 2.3 for  $S = \mathbb{Z}_{>0}$ .

Proposition 4.1 below implies that if  $|f| = \infty$  then  $\text{cls}(L(f)) = \mathcal{E}^\infty$ . One can check that the proof of Proposition 4.1 is independent from the current discussion. If  $|f| < \infty$ , there are two values  $a, b \in \mathbb{C}$  of  $f$  such that  $a - b \in \mathbb{Z}_{\geq 0}$  is maximal. We set  $s := a - b$ . For any nonnegative integer  $c \leq s$  we denote by  $d_c$  the multiplicity of the value  $b + c$  of  $f$  (note that  $d_c \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ ). Let  $p$  be the smallest integer such that  $d_p = +\infty$ , and  $q$  be the largest integer such that  $d_q = +\infty$  (if  $d_c$  is finite for all  $0 \leq c \leq b - a$ , we put  $p = q = 0$ ).

**Proposition 2.10.** a) For a  $\prec$ -dominant function  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  with  $|f| < \infty$ , we have

$$(4) \quad \text{cls}(L(f)) = \mathcal{L}_{d_0} \mathcal{L}_{d_0+d_1} \dots \mathcal{L}_{d_0+d_1+\dots+d_{p-1}} \mathcal{E}^{(q-p)} \mathcal{R}_{d_s} \mathcal{R}_{d_s+d_{s-1}} \dots \mathcal{R}_{d_s+\dots+d_{q+1}}.$$

b) A c.l.s. of the form (4) is of finite type.

c) Let  $\mathfrak{b}^0$  be a fixed ideal Borel subalgebra of  $\mathfrak{sl}(\infty)$ . Then any irreducible c.l.s. of finite type is equal to  $\text{cls}(L_{\mathfrak{b}^0}(f^0))$  for an appropriate  $\mathfrak{b}^0$ -dominant function  $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$ .

*Proof.* First we prove part a). Recall that  $L(f) = \varinjlim_{n \geq 2} L(f|_{\{1, \dots, n\}})$ . Thus the coherent local system  $\text{cls}(L(f))$  is

determined by the highest weights  $\lambda_n$  of the finite-dimensional  $\mathfrak{sl}(n)$ -modules  $L(f|_{\{1, \dots, n\}})$ . Such local systems have been considered by Zhilinskii [Zh1] and he provides an explicit algorithm which assigns to  $\{\lambda_i\}$  a c.l.s. of the form (4). This implies a).

b) It is clear that any c.l.s. of the form (4) is a c.l.s. of finite type.

c) The ideal subalgebra  $\mathfrak{b}^0$  defines a partition  $S_1 \sqcup S_2 \sqcup S_3$  of  $\mathbb{Z}_{>0}$  with fixed order preserving bijections  $\mathbb{Z}_{>0} \rightarrow S_1$ ,  $\mathbb{Z}_{<0} \rightarrow S_3$ . We denote the image of  $k \in \mathbb{Z}_{>0}$  in  $S_1$  by  $k^1$ , and the image of  $-k \in \mathbb{Z}_{<0}$  in  $S_3$  by  $k^3$ .

It is clear that any c.l.s.  $Q$  of the form (3) with  $v = w = 0$  can be presented in the form (4) for suitable integers  $p, q, d_0, d_1, \dots, d_{p-1}, d_s, d_{s-1}, \dots, d_{q+1}$ . Moreover,  $\text{cls}(L_{\mathfrak{b}^0}(f)) = Q$  for the  $\mathfrak{b}^0$ -dominant function  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$

defined as follows:

$$f(i) := \begin{cases} p & \text{if } i \in S_2 \text{ or } i = k^1 \text{ for } k > d_0 + \dots + d_{p-1}; \\ q & \text{if } i = k^3 \text{ for } k > d_s + \dots + d_{q+1}; \\ j & \text{if } i = k^1 \text{ and } d_0 + \dots + d_{j-1} < k \leq d_0 + \dots + d_j \text{ for } j < p; \\ j & \text{if } i = k^3 \text{ and } d_s + \dots + d_{j+1} < k \leq d_s + \dots + d_j \text{ for } j > q. \end{cases}$$

□

### 3. STATEMENTS OF MAIN RESULTS

**Theorem 3.1.** Let  $\mathfrak{b} \supset \mathfrak{h}$  be a splitting Borel subalgebra of  $\mathfrak{sl}(\infty)$ , and  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . Then  $\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$  if and only if  $f$  is almost integral and locally constant with respect to the linear order defined by  $\mathfrak{b}$ .

**Theorem 3.2.** The following conditions on a nonzero ideal  $I$  of  $U(\mathfrak{sl}(\infty))$  are equivalent:

- $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$  for some splitting Borel subalgebra  $\mathfrak{b} \supset \mathfrak{h}$  and some function  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ ;
- $I$  is a prime integrable ideal of  $U(\mathfrak{sl}(\infty))$ ;
- $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f^0)$  for some  $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$ , where  $\mathfrak{b}^0$  is any fixed ideal Borel subalgebra.

**Proposition 3.3.** If  $\mathfrak{b}$  is a nonideal Borel subalgebra then there exists a prime integrable ideal  $I$  which does not arise as the annihilator of a simple  $\mathfrak{b}$ -highest weight  $\mathfrak{sl}(\infty)$ -module.

We split the proof of Theorem 3.1 into two parts:

- a) if  $\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$ , then  $f$  is almost integral and locally constant;
- b) if  $f$  is almost integral and locally constant with respect to the order defined by  $\mathfrak{b}$ , then

$$\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0.$$

Parts a) and b) of Theorem 3.1 are proved in Sections 4 and 5 respectively. Theorem 3.2 and Proposition 3.3 are proved in Section 6.

### 4. PROOF OF THEOREM 3.1 A)

To prove Theorem 3.1 a), we fix a Borel subalgebra  $\mathfrak{b} \supset \mathfrak{h}$  of  $\mathfrak{sl}(\infty)$  and hence an order  $\prec$  on  $\mathbb{Z}_{>0}$ . Throughout Sections 4 and 5 we suppress the dependence from  $\mathfrak{b}$  and  $\prec$  in all notation. We set  $I(f) := \text{Ann}_{U(\mathfrak{sl}(\infty))} L(f)$  for any  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . Sometimes we consider the finite-dimensional Lie algebra  $\mathfrak{sl}(n)$ . In this case the fixed order  $\{1, \dots, n\}$  is the standard order, and  $I(f) \subset U(\mathfrak{sl}(n))$  is the annihilator of the simple  $\mathfrak{sl}(n)$ -module with highest weight  $\lambda_f$  for  $f \in \mathbb{C}^n$ .

Theorem 3.1a) follows from Propositions 4.1, 4.2 and 4.3 below.

**Proposition 4.1.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . If  $I(f)$  is nonzero, then  $|f| < \infty$ .

**Proposition 4.2.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . If  $I(f)$  is nonzero, then  $f$  is almost integral.

**Proposition 4.3.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . If  $I(f)$  is nonzero, then  $f$  is locally constant with respect to  $\prec$ .

We prove these propositions consecutively in Sections 4.1, 4.4 and 4.5. Clearly, Proposition 4.1 follows from Proposition 4.3, however we require Proposition 4.1 for the proof of Proposition 4.3. Propositions 4.2 and 4.3 rely on a version of the Robinson-Schensted algorithm which we present in Section 4.2.

**4.1. Proof of Proposition 4.1.** We start with some notation. Let  $n \geq 2$  be a positive integer. For any ideal  $I \subset U(\mathfrak{sl}(n))$  we denote by  $\text{gr}I \subset \mathbf{S}^*(\mathfrak{sl}(n))$  the associated graded ideal. By  $\text{Var}(I) \subset \mathfrak{sl}(n)^*$  we denote the set of zeros of  $\text{gr}I$ .

The radical ideals of the center  $ZU(\mathfrak{sl}(n))$  of  $U(\mathfrak{sl}(n))$  are in one-to-one correspondence with  $\mathfrak{S}_n$ -invariant closed subvarieties of  $\mathfrak{h}_n^*$ , where  $\mathfrak{h}_n$  is a fixed Cartan subalgebra of  $\mathfrak{sl}(n)$  and  $\mathfrak{S}_n$  is the symmetric group on  $n$  letters. Let  $I$  be an ideal of  $U(\mathfrak{sl}(n))$ . Then  $Z\text{Var}(I)$  denotes the subvariety of  $\mathfrak{h}_n^*$  corresponding to the radical of the ideal  $I \cap ZU(\mathfrak{sl}(n))$  of  $ZU(\mathfrak{sl}(n))$ . If  $\{I_t\}$  is any collection of ideals in  $U(\mathfrak{sl}(n))$ , then

$$(5) \quad Z\text{Var}(\cap_t I_t) = \overline{\cup_t Z\text{Var}(I_t)},$$

where here and below bar indicates Zariski closure.

Let  $\phi : \{1, \dots, n\} \rightarrow \mathbb{Z}_{>0}$  be an injective map. Slightly abusing notation, we denote by  $\phi$  the induced homomorphism

$$\phi : U(\mathfrak{sl}(n)) \rightarrow U(\mathfrak{sl}(\infty)).$$

By  $\text{inj}(n)$  we denote the set of injective maps from  $\{1, \dots, n\}$  to  $\mathbb{Z}_{>0}$ , and by  $\text{inj}_0(n)$  the set of order preserving maps from  $\{1, \dots, n\}$  to  $\mathbb{Z}_{>0}$  with respect to the standard order on  $\{1, \dots, n\}$  and the order  $\prec$  on  $\mathbb{Z}_{>0}$ .

By  $\mathfrak{sl}(\phi)$  we denote  $\mathfrak{sl}(\text{im } \phi) \subset \mathfrak{sl}(\infty)$ . For any  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  we set  $f_\phi := f \circ \phi$ . Then  $M(f_\phi) := M_{\mathfrak{b} \cap \mathfrak{sl}(\phi)}(f_\phi)$  and  $L(f_\phi) := L_{\mathfrak{b} \cap \mathfrak{sl}(\phi)}(f_\phi)$  are well defined  $(\mathfrak{b} \cap \mathfrak{sl}(\phi))$ -highest weight  $\mathfrak{sl}(\phi)$ -modules. If  $f$  is dominant and  $\phi \in \text{inj}_0(n)$ , then  $f_\phi$  is  $(\mathfrak{b} \cap \mathfrak{sl}(\phi))$ -dominant.

Let  $\phi \in \text{inj}_0(n)$  and  $\widetilde{M}(f)$  be any quotient of  $M(f)$ . It is well known that

$$\text{ZVar}(\text{Ann}_{\text{U}(\mathfrak{sl}(\phi))} M(f_\phi)) = \text{ZVar}(\text{Ann}_{\text{U}(\mathfrak{sl}(\phi))} \widetilde{M}(f_\phi)) = \text{ZVar}(\text{Ann}_{\text{U}(\mathfrak{sl}(\phi))} L(f_\phi)) = \mathfrak{G}_n(\rho_n + \lambda_{f_\phi}),$$

where  $\rho_n \subset \mathfrak{h}_n^*$  is the half-sum of positive roots.

Let  $\mathfrak{g}$  be a Lie algebra. The adjoint group of  $\mathfrak{g}$  is the subgroup of  $\text{Aut } \mathfrak{g}$  generated by the exponents of all nilpotent elements of  $\mathfrak{g}$ . We denote this group by  $\text{Adj } \mathfrak{g}$ .

**Lemma 4.4.** Let  $\phi_1 : \mathfrak{k} \rightarrow \mathfrak{g}$  and  $\phi_2 : \mathfrak{k} \rightarrow \mathfrak{g}$  be two  $\text{Adj } \mathfrak{g}$ -conjugate morphisms of Lie algebras. Let  $I$  be a two-sided ideal of  $\text{U}(\mathfrak{g})$ . Then

$$\phi_1^{-1}(I) = \phi_2^{-1}(I).$$

*Proof.* The adjoint action of  $\mathfrak{g}$  on  $\text{U}(\mathfrak{g})$  extends uniquely to an action of  $\text{Adj } \mathfrak{g}$  on  $\text{U}(\mathfrak{g})$ . The ideal  $I$  is  $\mathfrak{g}$ -stable and thus is  $\text{Adj } \mathfrak{g}$ -stable. Let  $g \in \text{Adj } \mathfrak{g}$  be such that  $\phi_1 = g \circ \phi_2$ . Then

$$\phi_1^{-1}(g(i)) = \phi_2^{-1}(i)$$

for any  $i \in I$ . Hence,

$$\phi_1^{-1}(I) = \phi_2^{-1}(I).$$

□

*Proof of Proposition 4.1.* Let  $I(f) \neq 0$ . Assume to the contrary that there exist  $i_1, \dots, i_s, \dots \in \mathbb{Z}_{>0}$  such that

$$f(i_1), \dots, f(i_s), \dots$$

are pairwise distinct elements of  $\mathbb{C}$ . As  $I(f) \neq \text{U}(\mathfrak{sl}(\infty))$ , there exists a positive integer  $n$  and an injective map  $\phi : \{1, \dots, n\} \rightarrow \mathbb{Z}_{>0}$  such that

$$I_\phi := I(f) \cap \text{U}(\mathfrak{sl}(\phi)) \neq 0,$$

or equivalently

$$(6) \quad \text{U}(\mathfrak{sl}(n)) \supset \phi^{-1}(I(f)) = \phi^{-1}(I_\phi) \neq 0.$$

Let  $\psi \in \text{inj}(n)$  be another map. Since  $\phi$  and  $\psi$  are conjugate via the adjoint group of  $\mathfrak{sl}(\infty)$ , we have

$$(7) \quad \phi^{-1}(I(f)) = \psi^{-1}(I(f)) \neq 0.$$

This means that  $\phi^{-1}(I(f))$  depends on  $n$  and  $f$  but not on  $\phi$ , and we set  $I_n := \phi^{-1}(I(f))$ .

Assume now that  $\phi \in \text{inj}_0(n)$ . Then the highest weight space of the  $\mathfrak{sl}(\infty)$ -module  $L(f)$  generates a highest weight  $\mathfrak{sl}(\phi)$ -submodule  $\widehat{L}(f_\phi)$ . Clearly,

$$\text{Ann}_{\text{U}(\mathfrak{sl}(\phi))} L(f) \subset \text{Ann}_{\text{U}(\mathfrak{sl}(\phi))} \widehat{L}(f_\phi).$$

Therefore,

$$I_n \subset \bigcap_{\phi \in \text{inj}_0(n)} \text{Ann}_{\text{U}(\mathfrak{sl}(n))} \widehat{L}(f_\phi)$$

and

$$I_n \cap \text{ZU}(\mathfrak{sl}(n)) \subset \bigcap_{\phi \in \text{inj}_0(n)} (\text{Ann}_{\text{U}(\mathfrak{sl}(n))} \widehat{L}(f_\phi) \cap \text{ZU}(\mathfrak{sl}(n))).$$

Hence, according to (5) we have

$$\overline{\bigcup_{\phi \in \text{inj}_0(n)} \mathfrak{G}_n(\rho_n + \lambda_{f_\phi})} = \mathfrak{G}_n(\rho_n + \overline{\bigcup_{\phi \in \text{inj}_0(n)} \lambda_{f_\phi}}) \subset \text{ZVar}(I_n).$$

We claim that

$$\overline{\mathfrak{G}_n(\bigcup_{\phi \in \text{inj}_0(n)} \lambda_{f_\phi})} = \mathfrak{h}_n^*,$$

and thus that

$$(8) \quad \text{ZVar}(I_n) = \mathfrak{h}_n^*.$$

Our claim is equivalent to the equality

$$\overline{\mathfrak{G}_n(\bigcup_{\phi \in \text{inj}_0(n)} \lambda_{f_\phi})} = \overline{(\bigcup_{\phi \in \text{inj}(n)} \lambda_{f_\phi})} = \mathfrak{h}_n^*$$



which is implied by the following equality:

$$(9) \quad \overline{(\cup_{\phi \in \text{inj}(n)} f_\phi)} = \mathbb{C}^n.$$

We now prove (9) by induction. The inclusion  $\{1, \dots, n-j\} \rightarrow \{1, \dots, n\}$  induces a restriction map

$$\text{res} : \mathbb{C}^n \rightarrow \mathbb{C}^{n-j}.$$

Denote by  $f_\psi^*$  the preimage of  $f_\psi$  under  $\text{res}$  for  $\psi \in \text{inj}(n-j)$ . We will show that

$$(10) \quad f_\psi^* \subset \overline{\cup_{\phi \in \text{inj}(n)} f_\phi}$$

for any  $j \leq n$  and any map  $\psi \in \text{inj}(n-j)$ . This holds trivially for  $j = 0$ . Assume that it also holds for  $j$ . Fix  $\psi \in \text{inj}(n-j-1)$  and set

$$(\psi \times k)(l) := \begin{cases} \psi(l) & \text{if } l \leq n-j-1 \\ i_k & \text{if } l = n-j \end{cases}.$$

It is clear that there exists  $s \in \mathbb{Z}_{\geq 1}$  such that

$$(\psi \times k) \in \text{inj}(n-j)$$

for any  $k \in \mathbb{Z}_{\geq s}$ . Moreover,  $f_{\psi \times k_1} \neq f_{\psi \times k_2}$  for any  $k_1 \neq k_2$ . Therefore

$$\overline{\cup_{k \in \mathbb{Z}_{\geq s}} f_{\psi \times k}} = f_{\psi}^*,$$

which yields (10).

For  $j = n$ , (10) yields  $\mathbb{C}^n \subset \overline{\cup_{\phi \in \text{inj}(n)} f_\phi}$ , consequently (9) holds. Then (8) holds also, hence

$$I_n \cap \text{ZU}(\mathfrak{sl}(n)) = 0.$$

It is a well known fact that an ideal of  $\text{U}(\mathfrak{sl}(n))$  whose intersection with  $\text{ZU}(\mathfrak{sl}(n))$  equals zero is the zero ideal [Dix, Proposition 4.2.2]. Therefore, we have a contradiction with (6), and the proof is complete.  $\square$

**4.2. Algorithm for  $\mathfrak{sl}(n)$ .** According to Duflo's Theorem, any primitive ideal of  $\text{U}(\mathfrak{sl}(n))$  is the annihilator of some simple highest weight module, i.e., any primitive ideal is of the form  $I(f)$  for some  $f \in \mathbb{C}^n$ . The associated variety of  $I(f)$  is the closure of a certain nilpotent coadjoint orbit  $\mathcal{O}(f)$  of  $\mathfrak{sl}(n)$  [Jo4]. To  $\mathcal{O}(f) \subset \mathfrak{sl}(n)^*$  one assigns a partition  $p(f)$  of  $n$  as follows. One first represents  $\mathcal{O}(f)$  by a nilpotent element  $x \in \mathfrak{sl}(n)$ . Then  $p(f)$  is the partition conjugate to the partition arising from the sizes of Jordan blocks of  $x$  considered as a linear operator on the natural representation of  $\mathfrak{sl}(n)$ .

We now describe the algorithm which computes  $p(f)$ . This is a modification of the Robinson-Schensted algorithm, see [Knu, Theorem A on p. 52].

Let  $f \in \mathbb{C}^n$  be a function.

Step 1) Set  $f^+ := (f(1), f(2) - 1, \dots, f(n) - n + 1)$ .

Step 2) Introduce an equivalence relation  $\sim$  on  $\{1, \dots, n\}$ :

$$i \sim j \text{ if and only if } f(i) - f(j) \in \mathbb{Z}.$$

Let  $t$  be the number of equivalence classes for  $\sim$ , and let  $n_1, \dots, n_t$  be the cardinalities of the respective equivalence classes.

Step 3) Consider  $f^+$  as a function  $f^+ : \{1, \dots, n\} \rightarrow \mathbb{C}$ . The restriction of  $f^+$  to the equivalence classes of Step 2) defines subsequences  $\text{seq}_1(f^+), \text{seq}_2(f^+), \dots, \text{seq}_t(f^+)$  of respective lengths  $n_1, \dots, n_t$ .

Step 4) Fix  $i$ . Note that the elements of  $\text{seq}_i(f^+)$  are linearly ordered as their pairwise differences are integers. Since the elements of  $\text{seq}_i(f^+)$  are not necessary pairwise distinct, we modify the above linear order by letting  $f^+(m) \triangleright f^+(k)$  if  $m > k$  and  $f^+(m) = f^+(k)$ . In this way we introduce a new linearly ordered set  $\widetilde{\text{seq}_i(f^+)}$  of cardinality  $n_i$ .

Step 5) Apply the Robinson-Schensted algorithm to the linearly ordered sets  $\widetilde{\text{seq}_i(f^+)}$  from Step 4) to produce partitions  $p_i$  of  $n_i$ .

Step 6) Consider the partitions  $p_1, p_2, \dots, p_t$  as a partition  $\text{RS}(f)$  of  $n$ .

**Proposition 4.5.** Let  $f \in \mathbb{C}^n$  be a function. Then  $p(f) = \text{RS}(f)$ .

*Proof.* This statement is contained in the work of A. Joseph, so all we need to do is to translate Joseph's result to the language which we use in this paper. For any  $f' \in \mathbb{C}^n$  set  $(f')^\# := (f'(1), f'(2) + 1, \dots, f'(n) + n - 1)$ .

We note first that

$$(11) \quad I(f) = I((\text{seq}_1(f^+), \dots, \text{seq}_t(f^+))^\#).$$

This is a translation of the equality

$$(12) \quad J(w_1 w_2 \lambda) = J(w_2 \lambda)$$

for appropriate choices of Weyl group elements  $w_1, w_2$ , as stated at the bottom of the first page of [Jo3] (the equality (12) uses the notation of A. Joseph which is slightly different from ours). Thus we can assume further that  $f^+ = (\text{seq}_1(f^+), \text{seq}_2(f^+), \dots, \text{seq}_t(f^+))$ .

Next, using the well known fact that  $p(f)$  is recovered uniquely from  $p(\text{seq}_i(f^+))$  for all  $i$ , we can suppose that  $f^+ = \text{seq}_1(f^+)$ , i.e., that  $f$  is integral.

In the case when  $f^+$  is regular, i.e. when  $f^+(k) \neq f^+(l)$  for  $k \neq l$ , Joseph states [Jo1, Section 3.3] that  $p(f)$  equals the shape of the output of the standard Robinson-Schensted algorithm [Knu, 5.1.4, proof of Theorem A] applied to the unique Weyl group element  $w$  such that  $w(f^+)$  is dominant. It is easy to check that this statement is equivalent to the claim that  $p(f) = \text{RS}(f)$  in this case.

In the case when  $f^+$  is not regular, following Joseph [Jo3, Section 2.1] we replace  $f$  by any function  $f'$  such that  $(f')^+$  is regular and  $f$  belongs to the upper closure  $\widehat{F}_{f'}$  of a certain facet  $F_{f'}$  containing  $f'$  [Jo3, Section 2.1]. In our language this means that  $f'$  and  $f$  satisfy the following conditions:

$$(13) \quad \text{if } f^+(i) > f^+(j), \text{ then } (f')^+(i) > (f')^+(j) \text{ for all } i, j \leq n;$$

$$(14) \quad \text{if } f^+(i) < f^+(j), \text{ then } (f')^+(i) < (f')^+(j) \text{ for all } i, j \leq n;$$

$$(15) \quad \text{if } f^+(i) = f^+(j) \text{ and } i < j, \text{ then } (f')^+(i) > (f')^+(j).$$

Then, according to Joseph,  $p(f) = p(f')$  [Jo2, Section 2.4]. A direct checking using (13)-(15) and the above linear order  $\triangleleft$  shows that in this case  $p(f') = \text{RS}(f)$ . □

**Example 4.6.** Let  $f = (\sqrt{2} - 1, 5, 9, \sqrt{2} + 3, 5, \sqrt{2} + 4, 7, 7) \in \mathbb{C}^8$ .

1)  $f^+ = (\sqrt{2} - 1, 4, 7, \sqrt{2}, 1, \sqrt{2} - 1, 1, 0)$ .

2-3)  $\widetilde{\text{seq}_1(f^+)} = (\sqrt{2} - 1, \sqrt{2}, \sqrt{2} - 1)$  ( $n_1 = 3$ ),  $\widetilde{\text{seq}_2(f^+)} = (4, 7, 1, 1, 0)$  ( $n_2 = 5$ ).

4)  $\text{seq}_1(f^+) = \{(\sqrt{2} - 1)', \sqrt{2}, (\sqrt{2} - 1)''\}$ ,  $\text{seq}_2(f^+) = \{4, 7, 1', 1'', 0\}$ .

5) Applying the Robinson-Schensted algorithm we have

$$\widetilde{\text{seq}_1(f^+)} \mapsto \begin{array}{|c|c|} \hline \sqrt{2} & (\sqrt{2} - 1)'' \\ \hline (\sqrt{2} - 1)' & \\ \hline \end{array} \mapsto (2, 1), \quad \widetilde{\text{seq}_2(f^+)} \mapsto \begin{array}{|c|c|c|} \hline 7 & 1'' & 0 \\ \hline 4 & 1' & \\ \hline \end{array} \mapsto (3, 2).$$

6)  $p(\sqrt{2} - 1, 5, 9, \sqrt{2} + 3, 5, \sqrt{2} + 4, 7, 7) = (2, 1) \cup (3, 2) = (3, 2, 2, 1)$ .

**4.3. Rank of a partition.** Let, as above,  $\mathcal{O}(f) \subset \mathfrak{sl}(n)^*$  be the nilpotent coadjoint orbit of  $\mathfrak{sl}(n)$  assigned to a function  $f \in \mathbb{C}^n$ . For  $x \in \mathcal{O}(f)$ , the rank of  $x$  is independent on  $x$  and equals  $n - p(f)_{\max}$ , where  $p(f)_{\max}$  is the maximal element of the partition  $p(f)$ . By definition, the integer  $p(f)_{\max}$  is the *corank* of  $p$ .

**Lemma 4.7.** Let  $f \in \mathbb{C}^n$ . The corank of  $p(f)$  equals the length of a longest strictly decreasing subsequence of  $f^+$  such that the difference between any two elements is an integer.

*Proof.* It is obvious that the corank of  $p(f)$  equals the maximum of coranks of  $p_1, \dots, p_t$ , where  $p_1, \dots, p_t$  are the partitions defined in Step 5) of Section 4.2. It is known that for each  $i$  the corank of  $p_i = \widetilde{p(\text{seq}_i(f^+))}$  equals to the length of a longest strictly decreasing subsequence [Knu, p. 69, Ex. 7] of  $\widetilde{\text{seq}_i(f^+)}$ . For some  $i_0$  a longest strictly decreasing subsequence of  $\widetilde{\text{seq}_{i_0}(f^+)}$  will also be a longest strictly decreasing subsequence of  $f^+$  such that the difference between any two elements is an integer, and the lemma is proved. □

**4.4. Proof of Proposition 4.2.** Proposition 4.2 is implied by the following two lemmas.

**Lemma 4.8.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . If  $I(f) \neq 0$ , there exists  $r \in \mathbb{Z}_{\geq 0}$  such that any finite subset  $F \subset \mathbb{Z}_{>0}$  has a subset  $F' \subset F$  so that  $f|_{F'}$  is integral and  $|F \setminus F'| \leq r$ .

**Lemma 4.9.** Fix  $r \in \mathbb{Z}_{\geq 0}$ . If for any finite subset  $F \subset \mathbb{Z}_{>0}$  there is  $F' \subset F$  so that  $f|_{F'}$  is integral and  $|F \setminus F'| \leq r$ , then there is a finite subset  $F \subset \mathbb{Z}_{>0}$  such that  $f|_{\mathbb{Z}_{>0} \setminus F}$  is integral and  $|F| \leq r$ .

**4.4.1. Proof of Lemma 4.8.** Due to the description of the corank of  $p(f)$  presented in Lemma 4.7, Lemma 4.8 is implied by the following lemma.

**Lemma 4.10.** Fix  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . If  $I(f) \neq 0$ , then there exists  $r \in \mathbb{Z}_{\geq 0}$  such that  $\text{rk } p(f|_F) \leq r$  for any finite subset  $F \subset \mathbb{Z}_{>0}$ .

*Proof.* Assuming that  $I(f) \neq 0$ , pick  $r$  as in Theorem 2.6. Let  $F$  be a finite subset of  $\mathbb{Z}_{>0}$ .

There is a nonzero homomorphism of  $\mathfrak{sl}(F)$ -modules  $M(f|_F) \rightarrow L(f)$ . Therefore, as  $L(f|_F)$  is the unique simple quotient of  $M(f|_F)$ ,  $L(f|_F)$  is isomorphic to a subquotient of  $L(f)$  considered as an  $\mathfrak{sl}(F)$ -module. This implies

$$(U(\mathfrak{sl}(F)) \cap I(f)) \cdot L(f|_F) = 0$$

and

$$\text{Var}(I(f) \cap U(\mathfrak{sl}(F))) \subset \mathfrak{sl}(F)^{\leq r}.$$

As all elements of  $\text{Var}(I(f|_F))$  are nilpotent, we have  $\text{rk } \mathcal{O}(f|_F) \leq r$ , and thus  $\text{rk } p(f|_F) \leq r$ .  $\square$

**4.4.2. Proof of Lemma 4.9.** We reduce the problem to a statement concerning the graph  $\Gamma := (\mathbb{Z}_{>0}, E_f)$  attached to the pair  $(\mathbb{Z}_{>0}, f)$  in the following way: the vertices of  $\Gamma$  are the elements of  $\mathbb{Z}_{>0}$ ,  $E_f$  stands for the edges of  $\Gamma$ , and  $i, j \in \mathbb{Z}_{>0}$  are connected by an edge if and only if  $f(i) - f(j) \notin \mathbb{Z}$ .

Lemma 4.9 is implied by the following lemma.

**Lemma 4.11.** Let  $\Gamma = (S, E)$  be a graph. Assume that there is  $r \in \mathbb{Z}_{\geq 0}$  so that any finite subset  $F \subset S$  decomposes into two subsets

$$\text{inf}(F) \cup \text{fin}(F)$$

with the properties

$$(16) \quad a) \Gamma|_{\text{inf}(F)} \text{ has no edges,} \quad b) |\text{fin}(F)| \leq r.$$

Then  $S$  decomposes into two subsets  $\text{inf}(S) \cup \text{fin}(S)$  satisfying (16) with  $F$  replaced by  $S$ .

*Proof.* In what follows we say that a vertex of  $S$  is connected with another vertex if they belong to a common edge. Denote by  $S^{>r}$  the set of vertices of  $S$  which belong to at least  $r+1$  edges. Respectively, let  $S^{\leq r}$  be the set of vertices of  $S$  which belong to at most  $r$  edges. In addition, denote by  $\underline{S}^{\leq r}$  the subset of  $S^{\leq r}$  consisting of vertices connected with at least one vertex from  $S^{\leq r}$ .

We claim that both  $S^{>r}$  and  $\underline{S}^{\leq r}$  are finite and

$$(17) \quad \text{i) } |S^{>r}| \leq r, \quad \text{ii) } |\underline{S}^{\leq r}| \leq r^2.$$

First we show (17) under the assumption that  $S^{>r}$  and  $\underline{S}^{\leq r}$  are finite. Let  $\tilde{S}^{>r}$  be a finite subset of  $S$  such that

$$1) S^{>r} \subset \tilde{S}^{>r},$$

$$2) \text{ any vertex from } S^{>r} \text{ is connected with at least } r+1 \text{ vertices from } \tilde{S}^{>r} \text{ (such a subset } \tilde{S}^{>r} \text{ always exists).}$$

A vertex  $i \in \text{inf}(\tilde{S}^{>r})$  can be connected only with vertices from  $\text{fin}(\tilde{S}^{>r})$ , and hence  $i \in S^{<r}$  by (16)a). Therefore,

$$\text{inf}(\tilde{S}^{>r}) \subset S^{\leq r} \cap \tilde{S}^{>r}.$$

This implies

$$(18) \quad S^{>r} \subset \text{fin}(\tilde{S}^{>r}),$$

and since  $|\text{fin}(\tilde{S}^{>r})| < r$  by (16)b), we obtain (17)i).

To prove (17)ii), note that since any vertex of  $\text{fin}(\underline{S}^{\leq r})$  belongs to at most  $r$  edges, the entire set  $\text{fin}(\underline{S}^{\leq r})$  belongs to at most  $r^2$  edges. As any vertex from  $\underline{S}^{\leq r}$  is connected with a vertex from  $\text{fin}(\underline{S}^{\leq r})$ , we obtain (17)ii).

Now we drop the assumption that both  $S^{>r}$  and  $\underline{S}^{\leq r}$  are finite. Applying the preceding arguments we show that (17) holds if we replace  $S^{>r}$  and  $\underline{S}^{\leq r}$  by their intersections with any finite subset of  $S$ . Thus (17) holds also for  $S^{>r}$  and  $\underline{S}^{\leq r}$ .

To finish the proof, we set

$$\text{fin}(S) := \text{fin}(\tilde{S}^{>r} \cup \underline{S}^{\leq r}).$$

Then  $|\text{fin}(S)| \leq r$  by (16)b). The same arguments by which we prove (18) imply

$$S^{>r} \subset \text{fin}(\tilde{S}^{>r} \cup \underline{S}^{\leq r}) := S.$$

Due to the definition of  $\underline{S}^{\leq r}$ , any vertex from

$$S \setminus (\tilde{S}^{>r} \cup \underline{S}^{\leq r})$$

can be connected only with vertices from  $S^{>r}$ . Thus  $\Gamma|_{S \setminus \text{fin}(S)}$  has no edges, and the proof is complete.  $\square$

#### 4.5. Proof of Proposition 4.3.

**Lemma 4.12.** Fix  $r \in \mathbb{Z}_{\geq 0}$ . Let  $f \in \mathbb{C}^{2r+2}$  be an integer valued function such that

$$(19) \quad f(2i) > f(2i-1)$$

for  $1 \leq i \leq r+1$ . Then  $\text{rk } p(f) > r$ .

*Proof.* Assume  $\text{rk } p(f) \leq r$ . Then the sequence  $f^+ = (f(1), f(2) - 1, \dots, f(n) - n + 1)$  contains a strictly decreasing subsequence  $\text{seq}'$  of length at least  $r+2$ . The set  $\{1, \dots, 2r+2\}$  is the disjoint union of  $r+1$  pairs of the form  $\{2i, 2i-1\}$ , hence for some  $i$  both  $f(2i-1) - (2i-1) + 1$  and  $f(2i) - 2i + 1$  belong to  $\text{seq}'$ . On the other hand,

$$f(2i-1) - (2i-1) + 1 \leq f(2i) - 2i + 1$$

by (19), thus  $\text{seq}'$  is not strictly decreasing. This contradiction shows that  $\text{rk } p(f) > r$ .  $\square$

*Proof of Proposition 4.3.* Assume that  $I(f) \neq 0$  and pick  $r$  as in Lemma 4.10. Using Proposition 4.1 and Proposition 4.2, we reduce Proposition 4.3 to the following statement:

If an integer valued function  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  takes finitely many values and there exists  $r \in \mathbb{Z}_{\geq 0}$  such that  $\text{rk } p(f|_F) \leq r$  for any finite subset  $F \subset \mathbb{Z}_{>0}$ , then  $f$  is locally constant.

We prove this statement by induction on  $|f|$ . The base of induction ( $|f| = 1$ ) is trivial.

Assume that the statement holds for  $|f| = n \geq 1$ , and let  $f$  be a function which takes precisely  $(n+1)$  values. Let  $M$  be the maximal value of  $f$ . Say that  $i, j \in \mathbb{Z}_{>0}, i \neq j$ , are equivalent whenever one of the following conditions hold:

- 1)  $i \prec j$ ,  $f(i) = f(j) = M$ , and  $f(s) = M$ , for any  $s, i \prec s \prec j$ ;
- 2)  $i \prec j$ ,  $f(i) < M$ ,  $f(j) < M$ , and  $f(s) < M$ , for any  $s, i \prec s \prec j$ .

It is easy to see that this is a well defined equivalence relation on  $\mathbb{Z}_{>0}$ . There are two possibilities for the respective equivalence classes  $S_\alpha$ :

- a)  $f(s) = M$  for any  $s \in S_\alpha$ ;
- b)  $f(s) < M$  for any  $s \in S_\alpha$ .

We claim that there exist no more than  $r+1$  equivalence classes of type b). Assume to the contrary that  $s_0 \prec s_2 \prec \dots \prec s_{2r+2}$  are elements from  $r+2$  distinct equivalence classes of type b). Then, for any  $i$ ,  $0 \leq i \leq r$ , there exists  $s_{2i+1} \in S$  such that

$$f(s_{2i+1}) = M \text{ and } s_{2i} \prec s_{2i+1} \prec s_{2i+2}.$$

The restriction of  $f$  to the set  $F := \{s_0, s_1, \dots, s_{2r+2}\}$  satisfies the assumption of Lemma 4.12. Hence  $\text{rk } p(f|_F) > r$ , which contradicts the statement of Lemma 4.10.

Therefore, there are at most  $r+1$  equivalence classes  $S_\alpha$  of type b). Any two classes of type a) must be separated by a class of type b), and hence there are at most  $r+2$  equivalence classes of type a). In particular the partition  $\sqcup_\alpha S_\alpha = \mathbb{Z}_{>0}$  is finite.

Clearly,  $f$  takes at most  $n$  values on each  $S_\alpha$ . By the induction assumption each  $S_\alpha$  admits a compatible partition such that  $f|_{S_\alpha}$  is locally constant. Therefore,  $f$  is also locally constant.  $\square$

### 5. PROOF OF THEOREM 3.1 B)

Theorem 3.1b) is a corollary of the following result.

**Proposition 5.1.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be a locally constant and almost integral function. Then there is a nonzero integrable ideal  $I$  of  $U(\mathfrak{sl}(\infty))$  such that  $I \subset I(f)$ .

We will prove a more precise version of this result. Let  $S_1 \sqcup \dots \sqcup S_t = \mathbb{Z}_{>0}$  be a fixed finite partition of  $\mathbb{Z}_{>0}$  compatible with the order  $\prec$ . Denote by  $S_{i_1}, \dots, S_{i_x}$  all infinite sets in this partition. By  $\gamma$  we denote the total number of elements in the finite sets of the partition. Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be a function locally constant with respect to the partition  $S_1 \sqcup \dots \sqcup S_t$ . It is easy to see that  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  is almost integral if and only if  $f(j) - f(k) \in \mathbb{Z}$  for any  $j \in S_{j'}$  and  $k \in S_{k'}$  such that both  $S_{j'}$  and  $S_{k'}$  are infinite. Under the assumption that  $f$  is almost integral, we set

$$(20) \quad \alpha(f) := \sum_{1 \leq j < x} \max(0, f(S_{i_{j+1}}) - f(S_{i_j})), \quad A(f) := \sum_{1 \leq j < x} \max(f(S_{i_j}) - f(S_{i_{j+1}}), 0),$$

where  $f(S_i)$  is the value of  $f$  on any element of  $S_i$  (we recall that  $f$  is constant on  $S_i$ ).

The following proposition is a more precise version of Proposition 5.1 and compares the annihilator of a simple highest weight module with the annihilator of a c.l.s. We will prove it by first establishing a finite-dimensional analogue, namely Proposition 5.3, and then showing that Proposition 5.2 actually reduces to this finite-dimensional analogue.

**Proposition 5.2.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be a function, locally constant with respect to the partition  $S_1 \sqcup \dots \sqcup S_t$  of  $\mathbb{Z}_{>0}$ . Then

$$I(\mathcal{L}_{(\alpha(f)+\gamma)}^\infty \mathcal{E}^{A(f)}) \subset I(f).$$

Let  $F$  be a finite subset of  $\mathbb{Z}_{>0}$ . Clearly,

$$(S_1 \cap F) \sqcup \dots \sqcup (S_t \cap F)$$

is a partition of  $F$ . We wish to define  $\alpha(f')$  and  $A(f')$  by formulas analogous to (20) for any function  $f' \in \mathbb{C}^F$  which is locally constant with respect to the partition  $(S_1 \cap F) \sqcup \dots \sqcup (S_t \cap F)$ . For this purpose we denote by  $S'_1$  the first  $S_{i_j}$  for which  $S_{i_j} \cap F \neq \emptyset$ , by  $S'_2$  the second  $S_{i_j}$  for which  $S_{i_j} \cap F \neq \emptyset$  and so on. Then we define  $\alpha(f')$  and  $A(f')$  by the respective right-hand sides of (20) applied to the subsets  $(S'_1 \cap F), (S'_2 \cap F), \dots$  instead of  $S_{i_1}, S_{i_2}, \dots$ . Finally,  $\gamma(F)$  stands for the total number of elements in all intersections  $S_i \cap F$  for finite sets  $S_i$ . For a large enough  $F$  we have  $\gamma(F) = \gamma, A(f|_F) = A(f), \alpha(f|_F) = \alpha(f)$ .

**Proposition 5.3.** Let  $F \subset \mathbb{Z}_{>0}$  be a finite subset with  $n$  elements, and  $f' \in \mathbb{C}^F$  be a function locally constant with respect to the partition  $(S_1 \cap F) \sqcup \dots \sqcup (S_t \cap F) = F$ . Then

$$I((\mathcal{L}_{(\alpha(f')+\gamma(F))}^\infty \mathcal{E}^{A(f')})_n) \subset I(f').$$

For the proof of Proposition 5.3 we need two lemmas (Lemmas 5.4 and 5.5 below) and some more notation. In Lemma 5.4  $f = (f_1, \dots, f_n)$  stands for a function  $f \in \mathbb{C}^n$ . We set  $L(f_1, \dots, f_n) := L(f)$  and  $I(f_1, \dots, f_n) := I(f)$  (where the fixed order on  $\{1, 2, \dots, n\}$  is the standard one). For a fixed nonnegative integer  $s < n$  and  $z_0 \in \mathbb{C}$ , we put:

$$\tilde{f} = (f_1, \dots, f_s, z_0, f_{s+1}, \dots, f_n) \in \mathbb{C}^{n+1}.$$

If  $A, B$  are two subsets of  $\text{Irr}_n$ ,  $A \otimes B$  stands for the set of isomorphism classes of all simple constituents of the tensor products  $\alpha \otimes \beta$  for  $\alpha \in A$  and  $\beta \in B$ .

**Lemma 5.4.** Let  $Q_n$  be a subset of  $\text{Irr}_n$  such that

$$I(Q_n) \subset I(f_1, \dots, f_s, f_{s+1} - 1, \dots, f_n - 1).$$

Then

$$I((\mathcal{L}_1^\infty)_n \otimes Q_n) \subset I(\tilde{f}),$$

$I(Q_n \otimes (\mathcal{L}_1^\infty)_n)$  being an ideal of  $U(\mathfrak{sl}(n))$  and  $I(\tilde{f})$  being an ideal of  $U(\mathfrak{sl}(n+1))$ .

*Proof.* Our idea is to replace  $z_0$  by a “generic value”. To do this, consider the supplementary Lie algebras

$$\mathfrak{sl}(n+1)[z] := \mathfrak{sl}(n+1) \otimes_{\mathbb{C}} \mathbb{C}[z] \subset \mathfrak{sl}(n+1)(z) := \mathfrak{sl}(n+1) \otimes_{\mathbb{C}} \overline{\mathbb{C}(z)},$$

the larger Lie algebra  $\mathfrak{sl}(n+1)(z)$  being finite dimensional and simple over the algebraically closed field  $\overline{\mathbb{C}(z)}$ . The sequence  $\hat{f} := (f_1, \dots, f_s, z, f_{s+1}, \dots, f_n)$  of elements of  $\overline{\mathbb{C}(z)}$  defines a weight  $\lambda_{\hat{f}} \in \mathfrak{h}_{n+1}^* \otimes \overline{\mathbb{C}(z)}$ .

Applying the equality (11) to  $\hat{f}$ , we obtain

$$I(\hat{f}) = I(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s).$$

By Proposition 2.2, we have

$$L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s) \cong U(\mathfrak{sl}(n+1)(z)) \otimes_{U(\mathfrak{p})} L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s)^{\mathfrak{n}},$$

where  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{sl}(n+1)(z)$  with a semisimple part  $\mathfrak{sl}(n)(z)$  and nilradical  $\mathfrak{n}$ . Proposition 2.2 yields also an isomorphism of  $\mathfrak{sl}(n)(z)$ -modules

$$L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s)^{\mathfrak{n}} \cong L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1) \otimes_{\mathbb{C}} \overline{\mathbb{C}(z)}.$$

Therefore we have an isomorphism of  $\mathfrak{sl}(n)$ -modules

$$L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s) \cong L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1) \otimes_{\mathbb{C}} \mathbf{S}(\overline{\mathbb{C}(z)})^n.$$

Hence  $L(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 1, \dots, f_n - 1, z + n - s)$  is annihilated by  $I(Q_n \otimes (\mathcal{L}_1^\infty)_n)$ , i.e.,

$$I(Q_n \otimes (\mathcal{L}_1^\infty)_n) \subset I(f_1, \dots, f_s, f_{s+1} - 1, f_{s+2} - 2, \dots, f_n - 1, z + n - s) = I(\hat{f}).$$

For this reason it suffices to show that

$$I(\hat{f}) \cap U(\mathfrak{sl}(n+1)) \subset I(\tilde{f})$$

for any  $z_0 \in \mathbb{C}$ .

Let  $v_{\hat{f}}$  be a highest weight vector of the  $\mathfrak{sl}(n+1)(z)$ -module  $L(\hat{f})$ . Consider the  $U(\mathfrak{sl}(n+1)[z])$ -module

$$(21) \quad U(\mathfrak{sl}(n+1)[z]) \cdot v_{\hat{f}}.$$

Clearly, the action of  $\mathfrak{h}_{n+1}$  on (21) is semisimple. The  $\lambda_{\tilde{f}}$ -weight space of (21) coincides with  $U(\mathfrak{h}_{n+1} \otimes \mathbb{C}[z]) \cdot v_{\tilde{f}}$ , and is isomorphic to  $\mathbb{C}[z]$  as a  $\mathbb{C}[z]$ -module. Therefore, the  $\lambda_{\tilde{f}}$ -weight space of the quotient

$$(22) \quad U(\mathfrak{sl}(n+1)[z]) \cdot v_{\tilde{f}} / (z - z_0) U(\mathfrak{sl}(n+1)[z]) \cdot v_{\tilde{f}}$$

is one-dimensional. In particular, the quotient (22) is nonzero.

Obviously, (22) is annihilated by

$$(23) \quad I(\tilde{f}) \cap U(\mathfrak{sl}(n+1)).$$

On the other hand, (22) has a highest weight vector of weight  $\lambda_{\tilde{f}}$ , and thus  $L(\tilde{f})$  is annihilated by (23). This is precisely what we have to prove.  $\square$

**Lemma 5.5.** Let  $F$  be a finite subset of  $\mathbb{Z}_{>0}$  with  $n$  elements, and  $f' \in \mathbb{C}^F$  be a function locally constant with respect to the partition  $(S_1 \cap F) \sqcup \dots \sqcup (S_t \cap F) = F$ . After identification of  $\mathfrak{sl}(F)$  with  $\mathfrak{sl}(n)$  we have

$$I((\mathcal{L}_{(\alpha(f')+\gamma(F))}^\infty \mathcal{E}^{A(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset I(f'),$$

where  $I((\mathcal{L}_{(\alpha(f')+\gamma(F))}^\infty \mathcal{E}^{A(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset U(\mathfrak{sl}(n - (\gamma(F) + \alpha(f'))))$  and  $I(f') \subset U(\mathfrak{sl}(n))$ .

*Proof.* We prove this lemma by induction on  $\gamma(F) + \alpha(f')$ .

Let  $\gamma(F) = \alpha(f') = 0$ . Then  $f'$  is dominant,  $L(f')$  is integrable and the statement of Lemma 5.4 follows from Lemma 2.10.

Next, assume that  $\gamma(F) + \alpha(f') = k + 1$  and that our statement holds for  $\gamma(F) + \alpha(f') \leq k$ . Then  $\alpha(f') > 0$  or  $\gamma(F) > 0$ . We consider both possibilities.

Let  $\alpha(f') > 0$ . Then  $f'(S'_j) < f'(S'_{j+1})$  for some  $j$ . Denote by  $s$  the maximal element of  $S'_j \cap F$  (with respect to the order inherited from the order  $\prec$ ). Put

$$F_- := F \setminus s, \quad f'_- := f'|_{F_-} \in \mathbb{C}^{F_-},$$

and note that  $f'_-$  is locally constant with respect to the partition

$$(S_1 \cap F_-) \sqcup \dots \sqcup (S_t \cap F_-)$$

of  $F_-$ . Moreover, it is easy to see that

$$\gamma(F_-) = \gamma(F), \quad \alpha(f'_-) < \alpha(f') \text{ and } A(f'_-) \leq A(f').$$

Thus we can apply the induction assumption to  $f'_-$ , which yields

$$I((\mathcal{L}_{\gamma(F)+\alpha(f'_-)}^\infty \otimes \mathcal{E}^{A(f'_-)})_{n-1-(\gamma(F)+\alpha(f'_-))}) \subset I(f'_-).$$

Applying Lemma 5.4 to  $s, z_0 = f'(s)$ , we obtain

$$(24) \quad I((\mathcal{L}_{\gamma(F)+\alpha(f'_-)+1}^\infty \mathcal{E}^{A(f'_-)})_{n-1-(\gamma(F)+\alpha(f'_-))}) \subset I(f').$$

Since

$$\gamma(F_-) + \alpha(f'_-) + 1 \leq \gamma(F) + \alpha(f') \text{ and } n - 1 - (\gamma(F_-) + \alpha(f'_-)) \geq n - (\gamma(F) + \alpha(f')),$$

(24) implies

$$I((\mathcal{L}_{\gamma(F)+\alpha(f')}^\infty \mathcal{E}^{A(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset I(f'),$$

which is precisely what we need to prove.

In the case when  $\alpha(f') = 0, \gamma(F) > 0$  we pick  $s$  to be the least element of  $F \setminus \bigcup_{j \leq x} S_{i_j}$  with respect to the order inherited from  $\prec$ . Then we apply the same arguments as above.  $\square$

**Remark 5.6.** It is clear that Lemma 5.5 applies to an arbitrary linearly ordered finite set  $F$ , an arbitrary compatible partition of  $F$ , an arbitrary function  $f \in \mathbb{C}^F$  locally constant with respect to this partition, and an arbitrary choice of equivalence classes of this partition used to define  $\alpha(\cdot), A(\cdot)$  and  $\gamma(\cdot)$ .

*Proof of Proposition 5.3.* Identify  $F$  with  $\{1, \dots, n\}$  as ordered sets (the order on  $F$  being inherited from the order  $\prec$ ). The function  $f' \in \mathbb{C}^F$  becomes  $f' = (f'_1, \dots, f'_n) \in \mathbb{C}^n$ . Let  $s$  be the least element of  $S'_1 \cap F$  under the above identification. Put

$$\tilde{f} := (f'_1, \dots, \underbrace{f'_s, f'_s, \dots, f'_s}_{(\gamma(F)+\alpha(f')+1) \text{ times}}, f'_{s+1}, \dots, f'_n).$$

It is clear that  $\tilde{f}$  is locally constant with respect to the partition  $\tilde{S}_1 \sqcup \tilde{S}_2 \sqcup \dots = \{1, \dots, n + \alpha(f') + \gamma(F)\}$ , which is defined as follows:

- (1)  $\tilde{S}_i$  coincides with  $(S_i \cap F)$  for  $i < j$ , where  $j$  is defined by the equality  $S_j = S'_1$ ;

- (2)  $\check{S}_j = (S_j \cap F) \cup \{\check{s} + 1, \dots, \check{s} + \alpha(f') + \gamma(F)\}$ , where  $\check{s}$  is the image in  $\{1, \dots, n\}$  of the last element of  $S_j \cap F$ ;  
 (3)  $\check{S}_i = \{\check{s}_- + \gamma(F) + \alpha(f'), \check{s}_- + \gamma(F) + \alpha(f') + 1, \dots, \check{s}_+ + \gamma(F) + \alpha(f') - 1, \check{s}_+ + \gamma(F) + \alpha(f')\}$  for  $i > j$ ,  
 where  $\check{s}_-$  and  $\check{s}_+$  are the images in  $\{1, \dots, n\}$  of the least and the greatest elements of  $S_i \cap F$ .

Remark 5.6 enables us to apply Lemma 5.5 to the function  $\check{f}$  and the partition  $\check{S}_1 \sqcup \check{S}_2 \sqcup \dots = \{1, \dots, n + \alpha(f') + \gamma(F)\}$ :

$$I((\mathcal{L}_{\gamma(F)+\alpha(f')}^\infty \mathcal{E}^{A(f')})_n) \subset I(\check{f}).$$

Finally, since  $L(f)$  is an  $\mathfrak{sl}(n)$ -subquotient of  $L(\check{f})$ , we have  $I(\check{f}) \cap U(\mathfrak{sl}(n)) \subset I(f')$ , and Proposition 5.3 is proved.  $\square$

Proposition 5.2 follows now from Proposition 5.3 and the next lemma.

**Lemma 5.7.** Let  $I \subset U(\mathfrak{sl}(\infty))$  be an ideal, and  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be a function. Then  $I \subset I(f)$  if and only if  $I_F := I \cap U(\mathfrak{sl}(F))$  annihilates  $L(f|_F)$  for any finite subset  $F$  of  $\mathbb{Z}_{>0}$ .

*Proof.* Let  $I \subset I(f)$ . Denote by  $v_f$  a highest weight vector of  $L(f)$ . If  $F$  is a finite set, then  $U(\mathfrak{sl}(F)) \cdot v_f$  is a highest weight  $\mathfrak{sl}(F)$ -submodule of  $L(f)$ . Thus  $L(f|_F)$  is isomorphic to a subquotient of  $L(f)$ , and consequently  $I_F = I \cap U(\mathfrak{sl}(F))$  annihilates  $L(f|_F)$ .

We now prove the converse. Set

$$M(F) := M(f|_F) / (I \cap U(\mathfrak{sl}(F)) \cdot M(f|_F)).$$

As  $I \cap U(\mathfrak{sl}(F))$  annihilates  $L(f|_F)$ ,  $M(F)$  is a nonzero highest weight  $\mathfrak{sl}(F)$ -module. Let  $v_f(F)$  be a highest weight vector of  $M(F)$ . For any finite subsets  $F_1 \subset F_2 \subset \mathbb{Z}_{>0}$ , there exists a unique morphism of  $\mathfrak{sl}(F_1)$ -modules

$$\phi_{F_1, F_2} : M(F_1) \rightarrow M(F_2)$$

such that  $\phi_{F_1, F_2}(v_f(F_1)) = v_f(F_2)$ . This defines a direct system of morphisms

$$\{\phi_{F_1, F_2}\}_{F_1 \subset F_2},$$

and we denote its limit by  $\tilde{M}(f)$ .

By definition,  $I$  annihilates the  $\mathfrak{sl}(\infty)$ -module  $\tilde{M}(f)$ . Our construction guarantees that  $\tilde{M}(f)$  contains a highest vector  $v_f := \varinjlim v_{f|_{F_i}}$  of weight  $\lambda_f$ . Thus  $L(f)$  is isomorphic to a simple quotient of  $\tilde{M}(f)$ , which implies  $I \subset I(f)$ .  $\square$

## 6. PROOF OF THEOREM 3.2 AND PROPOSITION 3.3

Theorem 3.2 is implied by the following propositions.

**Proposition 6.1.** Let  $\mathfrak{b} \supset \mathfrak{h}$  be a splitting Borel subalgebra of  $\mathfrak{sl}(\infty)$ , and  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  be function. Then  $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$  is a prime integrable ideal of  $U(\mathfrak{sl}(\infty))$ .

**Proposition 6.2.** Let  $I$  be a prime integrable ideal of  $U(\mathfrak{sl}(\infty))$  and  $\mathfrak{b}^0 \supset \mathfrak{h}$  be an ideal Borel subalgebra of  $\mathfrak{sl}(\infty)$ . Then  $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f^0)$  for some  $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$ .

**6.1. Proof of Proposition 6.1.** The annihilator of a simple module is always prime, therefore in order to prove Proposition 6.1 we have to prove that the ideal  $\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$  is integrable for any  $\mathfrak{b}$  and any  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ . This is a direct consequence of the following three statements.

**Proposition 6.3.** Let  $S$  be an infinite subset of  $\mathbb{Z}_{>0}$  and  $\phi : \mathbb{Z}_{>0} \rightarrow S$  be a fixed bijection. Let  $I$  be an ideal of  $U(\mathfrak{sl}(\infty))$ . Then the induced isomorphism  $\phi : U(\mathfrak{sl}(\infty)) \rightarrow U(\mathfrak{sl}(S))$  identifies  $I$  and  $I \cap U(\mathfrak{sl}(S))$ .

*Proof.* Fix the exhaustion (2) and assume that  $\mathfrak{sl}(n)$  is generated by  $e_{ij}$  for  $i \neq j, i, j \leq n$ . Then  $\mathfrak{sl}(S) = \cup_m \mathfrak{sl}(S_m)$ , where  $S_m$  is the image of  $\{1, \dots, m\}$  under  $\phi$ . We have

$$I \cap U(\mathfrak{sl}(S)) = \cup_m (I \cap U(\mathfrak{sl}(S_m))).$$

Since, for every  $n \geq 1$ ,  $\mathfrak{sl}(n)$  is  $\text{Adj } \mathfrak{sl}(\infty)$ -conjugate to  $\mathfrak{sl}(S_n)$ , Lemma 4.4 yields

$$\phi^{-1}(I \cap U(\mathfrak{sl}(S))) = \phi^{-1}(\cup_n (I \cap U(\mathfrak{sl}(S_n)))) = \cup_n (I \cap U(\mathfrak{sl}(n))) = I.$$

$\square$

**Corollary 6.4.** Let  $M$  be an  $\mathfrak{sl}(\infty)$ -module and  $S$  be an infinite subset of  $\mathbb{Z}_{>0}$ . Then  $\text{Ann}_{U(\mathfrak{sl}(\infty))} M$  is an integrable ideal in  $U(\mathfrak{sl}(\infty))$  if and only if  $\text{Ann}_{U(\mathfrak{sl}(S))} M$  is an integrable ideal of  $U(\mathfrak{sl}(S))$ .

**Proposition 6.5.** Let  $\mathfrak{b}$  and  $f$  be as in Proposition 6.1. If  $\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$ , then there exists an infinite subset  $S \subset \mathbb{Z}_{>0}$  such that  $L_{\mathfrak{b}}(f)$  is an integrable  $\mathfrak{sl}(S)$ -module.

*Proof.* As  $\text{Ann}_{\mathfrak{sl}(\infty)} L_{\mathfrak{b}}(f) \neq 0$ ,  $f$  is locally constant relative to some partition  $S_1 \sqcup \dots \sqcup S_t = \mathbb{Z}_{>0}$ , compatible with the order determined by  $\mathfrak{b}$ . Since  $S_i$  is infinite for some  $i$ , we apply Proposition 2.5 to conclude that  $L_{\mathfrak{b}}(f)$  is an integrable  $\mathfrak{sl}(S_i)$ -module.  $\square$

**6.2. Proof of Proposition 6.2.** Let  $\mathfrak{b}^0 \supset \mathfrak{h}$  be a fixed ideal Borel subalgebra of  $\mathfrak{sl}(\infty)$ . The goal of this section is to show that any integrable ideal is an annihilator of some  $\mathfrak{b}^0$ -highest weight module of  $\mathfrak{sl}(\infty)$ , and thus to prove Proposition 6.2. Due to the fact that an arbitrary irreducible c.l.s. has the form  $\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty$  for  $l, r \in \mathbb{Z}_{\geq 0}$  and some irreducible c.l.s. of finite type  $Q$ , it is enough to prove the following proposition.

**Proposition 6.6.** For any irreducible c.l.s.  $Q$  of finite type and any  $l, r \in \mathbb{Z}_{\geq 0}$  there exists  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  such that

$$\text{Ann}_{\mathfrak{sl}(\infty)} L_{\mathfrak{b}^0}(f) = I(\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty).$$

We fix  $l, r \in \mathbb{Z}_{\geq 0}$ . According to Proposition 6.3, the ideals  $\text{Ann}_{\mathfrak{sl}(\infty)} M$  and  $\text{Ann}_{\mathfrak{sl}(S)} M$  can be identified for any  $\mathfrak{sl}(\infty)$ -module  $M$  and any infinite subset  $S$  of  $\mathbb{Z}_{>0}$ . Therefore, Proposition 6.6 is implied by the following lemma.

**Lemma 6.7.** For any irreducible c.l.s.  $Q$  of finite type, there exist  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  and an infinite subset  $S \subset \mathbb{Z}_{>0}$  such that the  $\mathfrak{sl}(\infty)$ -module  $L_{\mathfrak{b}^0}(f)$  is integrable as an  $\mathfrak{sl}(S)$ -module and the c.l.s. for  $\mathfrak{sl}(S)$  of  $L_{\mathfrak{b}^0}(f)$  equals to  $\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty$ .

We now prove Lemma 6.7 by pointing out a concrete set  $S$  for which the claim of the lemma holds. We recall that the ideal Borel subalgebra  $\mathfrak{b}^0$  defines a partition  $S_1 \sqcup S_2 \sqcup S_3$  of  $\mathbb{Z}_{>0}$ . Let  $F_l$  be the set consisting of the first  $l$  elements of  $S_1$ . As an ordered set  $F_l$  is isomorphic to  $\{1, \dots, l\}$  with the standard order. Let  $F_r$  be set consisting of the the last  $r$  elements of  $S_3$ . As an ordered set  $F_r$  is isomorphic to  $\{-r, \dots, -1\}$  with the standard order. Put

$$S := \mathbb{Z}_{>0} \setminus (F_l \sqcup F_r).$$

Note that  $\mathfrak{b}_S^0 := \mathfrak{b}^0 \cap \mathfrak{sl}(S)$  is an ideal Borel subalgebra of  $\mathfrak{sl}(S)$ . Therefore, Proposition 2.10 c) asserts that, for any c.l.s.  $Q$  of finite type, there is a  $\mathfrak{b}_S^0$ -dominant function  $f^0 \in \mathbb{C}^S$  such that  $Q = \text{cls}(L_{\mathfrak{b}_S^0}(f^0))$ . For this reason Lemma 6.7 is a direct corollary of the following lemma.

**Lemma 6.8.** Let  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  satisfy the conditions

- 1)  $f|_S \in \mathbb{C}^S$  is  $\mathfrak{b}_S^0$ -dominant,
- 2)  $f(i) - f(j) \notin \mathbb{Z}$  for any  $i \neq j, j \in F_l$ ,
- 3)  $f(i) - f(j) \notin \mathbb{Z}$  for any  $i \neq j, j \in F_r$ .

Then the c.l.s. of the  $\mathfrak{sl}(S)$ -module  $L_{\mathfrak{b}^0}(f)$  is equal to  $\mathcal{L}_l^\infty \text{cls}(L_{\mathfrak{b}_S^0}(f|_S)) \mathcal{R}_r^\infty$ .

*Proof.* By Proposition 2.2,

$$L_{\mathfrak{b}^0}(f) \cong \mathfrak{U}(\mathfrak{sl}(\infty)) \otimes_{\mathfrak{U}(\mathfrak{p})} L_{\mathfrak{b}^0}(f)^{\mathfrak{n}},$$

where  $L_{\mathfrak{b}^0}(f)^{\mathfrak{n}} \cong L_{\mathfrak{b}_S^0}(f|_S)$  as an  $\mathfrak{sl}(S)$ -module. Hence, there is an isomorphism of  $\mathfrak{sl}(S)$ -modules

$$(25) \quad L_{\mathfrak{b}^0}(f) \cong \mathfrak{S}(\mathfrak{sl}(\infty)/\mathfrak{p}) \otimes_{\mathbb{C}} L_{\mathfrak{b}_S^0}(f|_S).$$

Furthermore, there is an isomorphism of  $\mathfrak{sl}(S)$ -modules

$$(26) \quad \mathfrak{sl}(\infty)/\mathfrak{p} \cong (V(S) \otimes \mathbb{C}^l \oplus \mathbb{C}^{\frac{l(l-1)}{2}}) \oplus (V(S)_* \otimes \mathbb{C}^r \oplus \mathbb{C}^{\frac{r(r-1)}{2}}) \oplus \mathbb{C}^{rl},$$

where  $V(S)$  is the natural  $\mathfrak{sl}(S)$ -module and  $\mathbb{C}$  stands for the one-dimensional trivial  $\mathfrak{sl}(S)$ -module. Thus,

$$(27) \quad \mathfrak{S}(\mathfrak{sl}(\infty)/\mathfrak{p}) \cong \mathfrak{S}(V(S) \otimes \mathbb{C}^l) \otimes \mathfrak{S}(V(S)_* \otimes \mathbb{C}^r) \otimes \mathfrak{S}(\mathbb{C}^{\frac{l(l-1)}{2}} \oplus \mathbb{C}^{\frac{r(r-1)}{2}}) \oplus \mathbb{C}^{rl}.$$

The c.l.s. of  $\mathfrak{S}(V(S) \otimes \mathbb{C}^l) = \mathfrak{S}(V(S))^{\otimes l}$  coincides with  $\mathcal{L}_l^\infty$ , and the c.l.s. of  $\mathfrak{S}(V(S)_* \otimes \mathbb{C}^r) = \mathfrak{S}(V(S)_*)^{\otimes r}$  coincides with  $\mathcal{R}_r^\infty$ . Hence the c.l.s. of  $\mathfrak{S}(\mathfrak{sl}(\infty)/\mathfrak{p})$  as an  $\mathfrak{sl}(S)$ -module coincides with  $\mathcal{L}_l^\infty \mathcal{R}_r^\infty$ , and the proof is complete.  $\square$

**Example 6.9.** Consider the fixed exhaustion (2) of  $\mathfrak{sl}(\infty)$ . Note that there is a canonical injection of  $\mathfrak{sl}(i)$ -modules  $\mathbf{S}^i V_i \rightarrow \mathbf{S}^{i+1} V_{i+1}$ , where  $V_i$  and  $V_{i+1}$  are respectively the natural representations of  $\mathfrak{sl}(i)$  and  $\mathfrak{sl}(i+1)$ . The direct limit  $D := \varinjlim_i \mathbf{S}^i V_i$  is a simple integrable  $\mathfrak{sl}(\infty)$ -module which is multiplicity free as an  $\mathfrak{h}$ -module. The module  $D$  has no highest weight with respect to any splitting Borel subalgebra  $\mathfrak{b}$  [DP1]. The c.l.s. corresponding to  $D$  equals  $\mathcal{L}_1^\infty$ , and in particular has infinite type. Lemma 6.8 implies that  $\text{Ann}_{\mathfrak{sl}(\infty)} D$  equals to the annihilator of a simple nonintegrable highest weight module. Indeed, let  $\mathfrak{b}^0$  be the ideal Borel subalgebra corresponding to the order (iii) in Section 2.1 and let  $f$  be the function

$$f(1) = \alpha \notin \mathbb{Z}, \quad f(n) = 0, \quad n > 1.$$



Then  $\text{Ann}_{U(\mathfrak{sl}(\infty))} D = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f)$ . This example illustrates the role of simple integrable non-highest weight modules in Theorem 3.2: the annihilators of such simple modules arise as annihilators of simple nonintegrable highest weight modules.

### 6.3. Proof of Proposition 3.3.

*Proof of Proposition 3.3.* We say that an ideal  $I$  of  $U(\mathfrak{sl}(\infty))$  is *of locally finite codimension* if  $I \cap U(\mathfrak{g})$  has finite codimension in  $U(\mathfrak{g})$  for any finite-dimensional subalgebra  $\mathfrak{g} \subset \mathfrak{sl}(\infty)$ . It is easy to see that such ideals have the following remarkable properties:

- (i) the map  $Q \mapsto I(Q)$  identifies the set of c.l.s. of finite type with the set of ideals of locally finite codimension;
- (ii) if an  $\mathfrak{sl}(\infty)$ -module  $M$  is annihilated by an ideal  $I \subset U(\mathfrak{sl}(\infty))$  of locally finite codimension, then  $M$  is integrable.

Using the properties (i) and (ii) one observes that if  $\mathfrak{b}$  is a Borel subalgebra, such that for any prime ideal  $I$  of locally finite codimension there exists  $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$  with  $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$ , then  $\mathfrak{b}$  is ideal. Indeed, Proposition 2.10a) gives an explicit expression of  $\text{cls}(L_{\mathfrak{b}}(f))$  in terms of  $f$ . The requirement that this procedure allows for every c.l.s. of finite type to appear in the right-hand side of (4) forces the existence of a  $\prec$ -compatible decomposition  $\mathbb{Z}_{>0} = F \sqcup S \sqcup F'$ , where  $F$  and  $F'$  are arbitrary finite sets. Clearly, this is equivalent to the requirement that  $\mathfrak{b}$  is ideal. □

## 7. ON SIMPLE $\mathfrak{sl}(\infty)$ -MODULES DETERMINED UP TO ISOMORPHISM BY THEIR ANNIHILATORS

It is a remarkable fact that if  $\mathfrak{g}$  is finite dimensional and semisimple, then a simple  $\mathfrak{g}$ -module  $M$  is determined up to isomorphism by its annihilator in  $U(\mathfrak{g})$  if and only if  $M$  is finite dimensional. We now provide an analogue of this fact for  $\mathfrak{sl}(\infty)$ .

Recall that a *simple tensor module* of  $\mathfrak{sl}(\infty)$  is a simple submodule of the tensor algebra

$$T^*(V(\infty) \oplus V(\infty)_*)$$

[DPS, PS]. It is easy to check that, for any fixed ideal Borel subalgebra  $\mathfrak{b}^0$ , the simple tensor modules are precisely the highest weight modules  $L_{\mathfrak{b}^0}(f)$  such that  $f$  can be chosen to be 0 almost everywhere (recall that the isomorphism class of a module  $L_{\mathfrak{b}^0}(f)$  recovers  $f$  up to an additive constant).

**Proposition 7.1.** Let  $M$  be a simple  $\mathfrak{sl}(\infty)$ -module which is determined up to isomorphism by its annihilator  $I = \text{Ann}_{U(\mathfrak{sl}(\infty))} M$ . If  $I$  is integrable, then  $M$  is isomorphic to a simple tensor module.

*Proof.* If  $I$  is not of locally finite codimension, then Lemma 6.8 shows that our assumption on  $M$  is contradictory as the function  $f$  from Lemma 6.8 is not determined uniquely by  $I$  up to an additive constant by  $I$ . In other words, if  $I$  is not of locally finite codimension, then Lemma 6.8 implies that there exist  $f_1, f_2 \in \mathbb{C}^{\mathbb{Z}_{>0}}$  such that

$$\text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f_1) = \text{Ann}_{U(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f_2) = I$$

but  $L_{\mathfrak{b}^0}(f_1) \not\cong L_{\mathfrak{b}^0}(f_2)$ .

Assume now that  $I$  has locally finite codimension. Then  $I = I(Q)$  for an irreducible c.l.s. of finite type  $Q$ , and by Proposition 2.10 c)  $M$  is isomorphic to  $L_{\mathfrak{b}^0}(f^0)$  for some ideal Borel subalgebra  $\mathfrak{b}^0$  and  $\mathfrak{b}^0$ -dominant function  $f^0$ . Moreover, as  $I$  is clearly fixed under the group  $\tilde{G} := \{g \in \text{Aut}_{\mathbb{C}} V(\infty) \mid g^*(V(\infty)_*) = V(\infty)_*\}$  considered as a group of automorphisms of  $U(\mathfrak{sl}(\infty))$ , it follows that  $M$  is invariant under  $\tilde{G}$ . Now Theorems 3.4 and 4.2 in [DPS] imply that  $L_{\mathfrak{b}}(f)$  is a simple tensor module.

It remains to show that a simple tensor  $\mathfrak{sl}(\infty)$ -module  $M$  is determined up to isomorphism by its annihilator  $\text{Ann}_{U(\mathfrak{sl}(\infty))} M$ . If  $M'$  is a simple  $\mathfrak{sl}(\infty)$ -module with  $\text{Ann}_{U(\mathfrak{sl}(\infty))} M' = \text{Ann}_{U(\mathfrak{sl}(\infty))} M = I$ , then the fact that  $I$  has locally finite codimension implies that  $M'$  is integrable and that the c.l.s. of  $M'$  coincides with the c.l.s. of  $I$ , i.e.,  $\text{cls}(M) = \text{cls}(M')$ . Furthermore, a careful analysis (carried out in detail in A. Sava's master's thesis [S]) shows that  $M'$  is a highest weight  $\mathfrak{sl}(\infty)$ -module with respect to the ideal Borel subalgebra given by the order (iii) of Section 2.1, and that the highest weight of  $M$  equals the highest weight of  $M'$ . This of course implies that  $M' \cong M$ . □

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